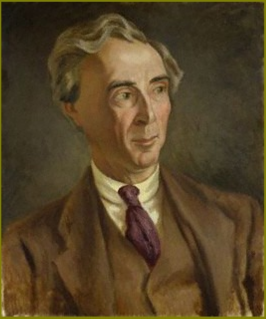
Introduction to Mathematical Philosophy



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Chapter XV

Propositional Functions

When, in the preceding chapter, we were discussing 155
propositions, we did not attempt to give a definition of the word "proposition." But although the word cannot be formally defined, it is necessary to say something as to its meaning, in order to avoid the very common confusion with "propositional functions," which are to be the topic of the present chapter.

We mean by a "proposition" primarily a form of words which expresses what is either true or false. I say "primarily," because I do not wish to exclude other than verbal symbols, or even mere thoughts if they have a symbolic character. But I think the word "proposition" should be limited to what may, in some sense, be called "symbols," and further to such symbols as give expression to truth and

falsehood. Thus "two and two are four" and "two
and two are five" will be propositions, and so will
"Socrates is a man" and "Socrates is not a man."
The statement: "Whatever numbers a and b may
be, $(a + b)^2 = a^2 + 2ab + b^2$" is a proposition; but
the bare formula "$(a + b)^2 = a^2 + 2ab + b^2$" alone is
not, since it asserts nothing definite unless we are
further told, or led to suppose, that a and b are to
have all possible values, or are to have such-and-
such values. The former of these is tacitly assumed,
as a rule, in the enunciation of mathematical for-
mulæ, which thus become propositions; but if no
such assumption were made, they would be "propo-
sitional functions." A "propositional function," in
fact, is an expression containing one or more un-
determined constituents, | such that, when values 156
are assigned to these constituents, the expression
becomes a proposition. In other words, it is a func-
tion whose values are propositions. But this latter
definition must be used with caution. A descrip-
tive function, *e.g.* "the hardest proposition in A's
mathematical treatise," will not be a propositional
function, although its values are propositions. But
in such a case the propositions are only described:
in a propositional function, the values must actually
enunciate propositions.

Examples of propositional functions are easy to give: "x is human" is a propositional function; so long as x remains undetermined, it is neither true nor false, but when a value is assigned to x it becomes a true or false proposition. Any mathematical equation is a propositional function. So long as the variables have no definite value, the equation is merely an expression awaiting determination in order to become a true or false proposition. If it is an equation containing one variable, it becomes true when the variable is made equal to a root of the equation, otherwise it becomes false; but if it is an "identity" it will be true when the variable is any number. The equation to a curve in a plane or to a surface in space is a propositional function, true for values of the co-ordinates belonging to points on the curve or surface, false for other values. Expressions of traditional logic such as "all A is B" are propositional functions: A and B have to be determined as definite classes before such expressions become true or false.

The notion of "cases" or "instances" depends upon propositional functions. Consider, for example, the kind of process suggested by what is called "generalisation," and let us take some very primitive example, say, "lightning is followed by thunder." We

have a number of "instances" of this, *i.e.* a number of propositions such as: "this is a flash of lightning and is followed by thunder." What are these occurrences "instances" of? They are instances of the propositional function: "If x is a flash of lightning, x is followed by thunder." The process of generalisation (with whose validity we are | fortunately not 157
concerned) consists in passing from a number of such instances to the *universal* truth of the propositional function: "If x is a flash of lightning, x is followed by thunder." It will be found that, in an analogous way, propositional functions are always involved whenever we talk of instances or cases or examples.

We do not need to ask, or attempt to answer, the question: "What *is* a propositional function?" A propositional function standing all alone may be taken to be a mere schema, a mere shell, an empty receptacle for meaning, not something already significant. We are concerned with propositional functions, broadly speaking, in two ways: first, as involved in the notions "true in all cases" and "true in some cases"; secondly, as involved in the theory of classes and relations. The second of these topics we will postpone to a later chapter; the first must occupy us now.

When we say that something is "always true" or "true in all cases," it is clear that the "something" involved cannot be a proposition. A proposition is just true or false, and there is an end of the matter. There are no instances or cases of "Socrates is a man" or "Napoleon died at St Helena." These are propositions, and it would be meaningless to speak of their being true "in all cases." This phrase is only applicable to propositional *functions*. Take, for example, the sort of thing that is often said when causation is being discussed. (We are not concerned with the truth or falsehood of what is said, but only with its logical analysis.) We are told that A is, in every instance, followed by B. Now if there are "instances" of A, A must be some general concept of which it is significant to say "x_1 is A," "x_2 is A," "x_3 is A," and so on, where x_1, x_2, x_3 are particulars which are not identical one with another. This applies, *e.g.*, to our previous case of lightning. We say that lightning (A) is followed by thunder (B). But the separate flashes are particulars, not identical, but sharing the common property of being lightning. The only way of expressing a | common property generally is to say 158
that a common property of a number of objects is a propositional function which becomes true when any one of these objects is taken as the value of the

variable. In this case all the objects are "instances" of the truth of the propositional function—for a propositional function, though it cannot itself be true or false, is true in certain instances and false in certain others, unless it is "always true" or "always false." When, to return to our example, we say that A is in every instance followed by B, we mean that, whatever x may be, if x is an A, it is followed by a B; that is, we are asserting that a certain propositional function is "always true."

Sentences involving such words as "all," "every," "a," "the," "some" require propositional functions for their interpretation. The way in which propositional functions occur can be explained by means of two of the above words, namely, "all" and "some."

There are, in the last analysis, only two things that can be done with a propositional function: one is to assert that it is true in *all* cases, the other to assert that it is true in at least one case, or in *some* cases (as we shall say, assuming that there is to be no necessary implication of a plurality of cases). All the other uses of propositional functions can be reduced to these two. When we say that a propositional function is true "in all cases," or "always" (as we shall also say, without any temporal suggestion), we mean that all its values are true. If "ϕx" is the

function, and a is the right sort of object to be an argument to "ϕx," then ϕa is to be true, however a may have been chosen. For example, "if a is human, a is mortal" is true whether a is human or not; in fact, every proposition of this form is true. Thus the propositional function "if x is human, x is mortal" is "always true," or "true in all cases." Or, again, the statement "there are no unicorns" is the same as the statement "the propositional function 'x is not a unicorn' is true in all cases." The assertions in the preceding chapter about propositions, *e.g.* "'p or q' implies 'q or p,'" are really assertions | that certain 159
propositional functions are true in all cases. We do not assert the above principle, for example, as being true only of this or that particular p or q, but as being true of *any* p or q concerning which it can be made significantly. The condition that a function is to be *significant* for a given argument is the same as the condition that it shall have a value for that argument, either true or false. The study of the conditions of significance belongs to the doctrine of types, which we shall not pursue beyond the sketch given in the preceding chapter.

Not only the principles of deduction, but all the primitive propositions of logic, consist of assertions that certain propositional functions are always true.

If this were not the case, they would have to mention particular things or concepts—Socrates, or redness, or east and west, or what not—and clearly it is not the province of logic to make assertions which are true concerning one such thing or concept but not concerning another. It is part of the definition of logic (but not the whole of its definition) that all its propositions are completely general, *i.e.* they all consist of the assertion that some propositional function containing no constant terms is always true. We shall return in our final chapter to the discussion of propositional functions containing no constant terms. For the present we will proceed to the other thing that is to be done with a propositional function, namely, the assertion that it is "sometimes true," *i.e.* true in at least one instance.

When we say "there are men," that means that the propositional function "x is a man" is sometimes true. When we say "some men are Greeks," that means that the propositional function "x is a man and a Greek" is sometimes true. When we say "cannibals still exist in Africa," that means that the propositional function "x is a cannibal now in Africa" is sometimes true, *i.e.* is true for some values of x. To say "there are at least n individuals in the world" is to say that the propositional function "α is

a class of individuals and a member of the cardinal number n" is sometimes true, or, as we may say, is true for certain | values of α. This form of expres- 160
sion is more convenient when it is necessary to indicate which is the variable constituent which we are taking as the argument to our propositional function. For example, the above propositional function, which we may shorten to "α is a class of n individuals," contains two variables, α and n. The axiom of infinity, in the language of propositional functions, is: "The propositional function 'if n is an inductive number, it is true for some values of α that α is a class of n individuals' is true for all possible values of n." Here there is a subordinate function, "α is a class of n individuals," which is said to be, in respect of α, *sometimes* true; and the assertion that this happens if n is an inductive number is said to be, in respect of n, *always* true.

The statement that a function ϕx is always true is the negation of the statement that not-ϕx is sometimes true, and the statement that ϕx is sometimes true is the negation of the statement that not-ϕx is always true. Thus the statement "all men are mortals" is the negation of the statement that the function "x is an immortal man" is sometimes true. And the statement "there are unicorns" is the negation of

the statement that the function "x is not a unicorn" is always true.[1] We say that ϕx is "never true" or "always false" if not-ϕx is always true. We can, if we choose, take one of the pair "always," "sometimes" as a primitive idea, and define the other by means of the one and negation. Thus if we choose "sometimes" as our primitive idea, we can define: "'ϕx is always true' is to mean 'it is false that not-ϕx is sometimes true.'" But for reasons connected with the theory of types it seems more correct to take both "always" and "sometimes" as primitive ideas, and define by their means the negation of propositions in which they occur. That is to say, assuming that we have already | defined (or adopted 161
as a primitive idea) the negation of propositions of the type to which ϕx belongs, we define: "The negation of 'ϕx always' is 'not-ϕx sometimes'; and the negation of 'ϕx sometimes' is 'not-ϕx always.'" In like manner we can re-define disjunction and the other truth-functions, as applied to propositions containing apparent variables, in terms of the definitions and primitive ideas for propositions containing no apparent variables. Propositions containing

[1] For linguistic reasons, to avoid suggesting either the plural or the singular, it is often convenient to say "ϕx is not always false" rather than "ϕx sometimes" or "ϕx is sometimes true."

no apparent variables are called "elementary propositions." From these we can mount up step by step, using such methods as have just been indicated, to the theory of truth-functions as applied to propositions containing one, two, three ... variables, or any number up to n, where n is any assigned finite number.[2]

The forms which are taken as simplest in traditional formal logic are really far from being so, and all involve the assertion of all values or some values of a compound propositional function. Take, to begin with, "all S is P." We will take it that S is defined by a propositional function ϕx, and P by a propositional function ψx. *E.g.*, if S is *men*, ϕx will be "x is human"; if P is *mortals*, ψx will be "there is a time at which x dies." Then "all S is P" means: "'ϕx implies ψx' is always true." It is to be observed that "all S is P" does not apply only to those terms that actually are S's; it says something equally about terms which are not S's. Suppose we come across an x of which we do not know whether it is an S or not; still, our statement "all S is P" tells us something about x, namely, that *if* x is an S, then x is a P. And this is every bit as true when x is not an S as

[2] The method of deduction is given in *Principia Mathematica*, vol. i. ∗9.

when *x* is an S. If it were not equally true in both cases, the *reductio ad absurdum* would not be a valid method; for the essence of this method consists in using implications in cases where (as it afterwards turns out) the hypothesis is false. We may put the matter another way. In order to understand "all S is P," it is not necessary to be able to enumerate what terms are S's; provided we know what is meant by being an S and what by being a P, we can understand completely what is actually affirmed | by "all S is P," 162
however little we may know of actual instances of either. This shows that it is not merely the actual terms that are S's that are relevant in the statement "all S is P," but all the terms concerning which the supposition that they are S's is significant, *i.e.* all the terms that are S's, together with all the terms that are not S's—*i.e.* the whole of the appropriate logical "type." What applies to statements about *all* applies also to statements about *some*. "There are men," *e.g.*, means that "*x* is human" is true for *some* values of *x*. Here *all* values of *x* (*i.e.* all values for which "*x* is human" is significant, whether true or false) are relevant, and not only those that in fact are human. (This becomes obvious if we consider how we could prove such a statement to be *false*.) Every assertion about "all" or "some" thus involves

not only the arguments that make a certain function true, but all that make it significant, *i.e.* all for which it has a value at all, whether true or false.

We may now proceed with our interpretation of the traditional forms of the old-fashioned formal logic. We assume that S is those terms x for which ϕx is true, and P is those for which ψx is true. (As we shall see in a later chapter, all classes are derived in this way from propositional functions.) Then:

"All S is P" means "'ϕx implies ψx' is always true."
"Some S is P" means "'ϕx and ψx' is sometimes true."
"No S is P" means "'ϕx implies not-ψx' is always true."
"Some S is not P" means "'ϕx and not-ψx' is sometimes true."

It will be observed that the propositional functions which are here asserted for all or some values are not ϕx and ψx themselves, but truth-functions of ϕx and ψx for the *same* argument x. The easiest way to conceive of the sort of thing that is intended is to start not from ϕx and ψx in general, but from ϕa and ψa, where a is some constant. Suppose we are considering "all men are mortal": we will begin with

"If Socrates is human, Socrates is mortal,"

and then we will regard "Socrates" as replaced by 163
a variable x wherever "Socrates" occurs. The object to be secured is that, although x remains a variable, without any definite value, yet it is to have the same value in "ϕx" as in "ψx" when we are asserting that "ϕx implies ψx" is always true. This requires that we shall start with a function whose values are such as "ϕa implies ψa," rather than with two separate functions ϕx and ψx; for if we start with two separate functions we can never secure that the x, while remaining undetermined, shall have the same value in both.

For brevity we say "ϕx always implies ψx" when we mean that "ϕx implies ψx" is always true. Propositions of the form "ϕx always implies ψx" are called "formal implications"; this name is given equally if there are several variables.

The above definitions show how far removed from the simplest forms are such propositions as "all S is P," with which traditional logic begins. It is typical of the lack of analysis involved that traditional logic treats "all S is P" as a proposition of the same form as "x is P"—*e.g.*, it treats "all men are mortal" as of the same form as "Socrates is mortal." As we have just seen, the first is of the form "ϕx

always implies ψx," while the second is of the form "ψx." The emphatic separation of these two forms, which was effected by Peano and Frege, was a very vital advance in symbolic logic.

It will be seen that "all S is P" and "no S is P" do not really differ in form, except by the substitution of not-ψx for ψx, and that the same applies to "some S is P" and "some S is not P." It should also be observed that the traditional rules of conversion are faulty, if we adopt the view, which is the only technically tolerable one, that such propositions as "all S is P" do not involve the "existence" of S's, *i.e.* do not require that there should be terms which are S's. The above definitions lead to the result that, if ϕx is always false, *i.e.* if there are no S's, then "all S is P" and "no S is P" will both be true, | whatever P 164
may be. For, according to the definition in the last chapter, "ϕx implies ψx" means "not-ϕx or ψx," which is always true if not-ϕx is always true. At the first moment, this result might lead the reader to desire different definitions, but a little practical experience soon shows that any different definitions would be inconvenient and would conceal the important ideas. The proposition "ϕx always implies ψx, and ϕx is sometimes true" is essentially composite, and it would be very awkward to give this as

the definition of "all S is P," for then we should have no language left for "ϕx always implies ψx," which is needed a hundred times for once that the other is needed. But, with our definitions, "all S is P" does not imply "some S is P," since the first allows the non-existence of S and the second does not; thus conversion *per accidens* becomes invalid, and some moods of the syllogism are fallacious, *e.g.* Darapti: "All M is S, all M is P, therefore some S is P," which fails if there is no M.

The notion of "existence" has several forms, one of which will occupy us in the next chapter; but the fundamental form is that which is derived immediately from the notion of "sometimes true." We say that an argument a "satisfies" a function ϕx if ϕa is true; this is the same sense in which the roots of an equation are said to satisfy the equation. Now if ϕx is sometimes true, we may say there are x's for which it is true, or we may say "arguments satisfying ϕx *exist*." This is the fundamental meaning of the word "existence." Other meanings are either derived from this, or embody mere confusion of thought. We may correctly say "men exist," meaning that "x is a man" is sometimes true. But if we make a pseudo-syllogism: "Men exist, Socrates is a man, therefore Socrates exists," we are talking non-

sense, since "Socrates" is not, like "men," merely an undetermined argument to a given propositional function. The fallacy is closely analogous to that of the argument: "Men are numerous, Socrates is a man, therefore Socrates is numerous." In this case it is obvious that the conclusion is nonsensical,
but | in the case of existence it is not obvious, for 165
reasons which will appear more fully in the next chapter. For the present let us merely note the fact that, though it is correct to say "men exist," it is incorrect, or rather meaningless, to ascribe existence to a given particular x who happens to be a man. Generally, "terms satisfying ϕx exist" means "ϕx is sometimes true"; but "a exists" (where a is a term satisfying ϕx) is a mere noise or shape, devoid of significance. It will be found that by bearing in mind this simple fallacy we can solve many ancient philosophical puzzles concerning the meaning of existence.

Another set of notions as to which philosophy has allowed itself to fall into hopeless confusions through not sufficiently separating propositions and propositional functions are the notions of "modality": *necessary*, *possible*, and *impossible*. (Sometimes *contingent* or *assertoric* is used instead of *possible*.) The traditional view was that, among true propo-

sitions, some were necessary, while others were merely contingent or assertoric; while among false propositions some were impossible, namely, those whose contradictories were necessary, while others merely happened not to be true. In fact, however, there was never any clear account of what was added to truth by the conception of necessity. In the case of propositional functions, the threefold division is obvious. If "ϕx" is an undetermined value of a certain propositional function, it will be *necessary* if the function is always true, *possible* if it is sometimes true, and *impossible* if it is never true. This sort of situation arises in regard to probability, for example. Suppose a ball x is drawn from a bag which contains a number of balls: if all the balls are white, "x is white" is necessary; if some are white, it is possible; if none, it is impossible. Here all that is *known* about x is that it satisfies a certain propositional function, namely, "x was a ball in the bag." This is a situation which is general in probability problems and not uncommon in practical life—*e.g.* when a person calls of whom we know nothing except that he brings a letter of introduction from our
friend so-and-so. In all such cases, as in regard to 166
modality in general, the propositional function is relevant. For clear thinking, in many very diverse

directions, the habit of keeping propositional functions sharply separated from propositions is of the utmost importance, and the failure to do so in the past has been a disgrace to philosophy.

Chapter XVI

Descriptions

We dealt in the preceding chapter with the words 167
all and *some*; in this chapter we shall consider the word *the* in the singular, and in the next chapter we shall consider the word *the* in the plural. It may be thought excessive to devote two chapters to one word, but to the philosophical mathematician it is a word of very great importance: like Browning's Grammarian with the enclitic $\delta\epsilon$, I would give the doctrine of this word if I were "dead from the waist down" and not merely in a prison.

We have already had occasion to mention "descriptive functions," *i.e.* such expressions as "the father of x" or "the sine of x." These are to be defined by first defining "descriptions."

A "description" may be of two sorts, definite and indefinite (or ambiguous). An indefinite description is a phrase of the form "a so-and-so," and a definite description is a phrase of the form "the so-and-so" (in the singular). Let us begin with the former.

"Who did you meet?" "I met a man." "That is a very indefinite description." We are therefore not departing from usage in our terminology. Our question is: What do I really assert when I assert "I met a man"? Let us assume, for the moment, that my assertion is true, and that in fact I met Jones. It is clear that what I assert is *not* "I met Jones." I may say "I met a man, but it was not Jones"; in that case, though I lie, I do not contradict myself, as I should do if when I say I met a | man I really mean that I 168
met Jones. It is clear also that the person to whom I am speaking can understand what I say, even if he is a foreigner and has never heard of Jones.

But we may go further: not only Jones, but no actual man, enters into my statement. This becomes obvious when the statement is false, since then there is no more reason why Jones should be supposed to enter into the proposition than why anyone else should. Indeed the statement would remain significant, though it could not possibly be true, even if there were no man at all. "I met a unicorn" or "I met a sea-serpent" is a perfectly significant assertion, if we know what it would be to be a unicorn or a sea-serpent, *i.e.* what is the definition of these fabulous monsters. Thus it is only what we may call the *concept* that enters into the proposition. In the case of "unicorn," for example, there is only the concept: there is not also, somewhere among the shades, something unreal which may be called "a unicorn." Therefore, since it is significant (though false) to say "I met a unicorn," it is clear that this proposition, rightly analysed, does not contain a constituent "a unicorn," though it does contain the concept "unicorn."

The question of "unreality," which confronts us at this point, is a very important one. Misled by grammar, the great majority of those logicians who have dealt with this question have dealt with it on mistaken lines. They have regarded grammatical form as a surer guide in analysis than, in fact, it is. And they have not known what differences in grammatical form are important. "I met Jones" and "I met a man" would count traditionally as propositions of the same form, but in actual fact they are of quite different forms: the first names an actual person, Jones; while the second involves a propositional function, and becomes, when made explicit: "The function 'I met x and x is human' is sometimes true." (It will be remembered that we adopted the

convention of using "sometimes" as not implying more than once.) This proposition is obviously not of the form "I met *x*," which accounts | for the exis- 169
tence of the proposition "I met a unicorn" in spite of the fact that there is no such thing as "a unicorn."

For want of the apparatus of propositional functions, many logicians have been driven to the conclusion that there are unreal objects. It is argued, *e.g.* by Meinong,[1] that we can speak about "the golden mountain," "the round square," and so on; we can make true propositions of which these are the subjects; hence they must have some kind of logical being, since otherwise the propositions in which they occur would be meaningless. In such theories, it seems to me, there is a failure of that feeling for reality which ought to be preserved even in the most abstract studies. Logic, I should maintain, must no more admit a unicorn than zoology can; for logic is concerned with the real world just as truly as zoology, though with its more abstract and general features. To say that unicorns have an existence in heraldry, or in literature, or in imagination, is a most pitiful and paltry evasion. What exists in heraldry is not an animal, made of flesh and blood, moving and breathing of its own initiative. What exists

[1] *Untersuchungen zur Gegenstandstheorie und Psychologie,* 1904.

is a picture, or a description in words. Similarly,
to maintain that Hamlet, for example, exists in his
own world, namely, in the world of Shakespeare's
imagination, just as truly as (say) Napoleon existed
in the ordinary world, is to say something deliber-
ately confusing, or else confused to a degree which
is scarcely credible. There is only one world, the
"real" world: Shakespeare's imagination is part of
it, and the thoughts that he had in writing Hamlet
are real. So are the thoughts that we have in reading
the play. But it is of the very essence of fiction that
only the thoughts, feelings, etc., in Shakespeare and
his readers are real, and that there is not, in addition
to them, an objective Hamlet. When you have taken
account of all the feelings roused by Napoleon in
writers and readers of history, you have not touched
the actual man; but in the case of Hamlet you have
come to the end of him. If no one thought about
Hamlet, there would be nothing | left of him; if no 170
one had thought about Napoleon, he would have
soon seen to it that some one did. The sense of real-
ity is vital in logic, and whoever juggles with it by
pretending that Hamlet has another kind of reality
is doing a disservice to thought. A robust sense of
reality is very necessary in framing a correct analysis
of propositions about unicorns, golden mountains,

round squares, and other such pseudo-objects.

In obedience to the feeling of reality, we shall insist that, in the analysis of propositions, nothing "unreal" is to be admitted. But, after all, if there *is* nothing unreal, how, it may be asked, *could* we admit anything unreal? The reply is that, in dealing with propositions, we are dealing in the first instance with symbols, and if we attribute significance to groups of symbols which have no significance, we shall fall into the error of admitting unrealities, in the only sense in which this is possible, namely, as objects described. In the proposition "I met a unicorn," the whole four words together make a significant proposition, and the word "unicorn" by itself is significant, in just the same sense as the word "man." But the *two* words "a unicorn" do not form a subordinate group having a meaning of its own. Thus if we falsely attribute meaning to these two words, we find ourselves saddled with "a unicorn," and with the problem how there can be such a thing in a world where there are no unicorns. "A unicorn" is an indefinite description which describes nothing. It is not an indefinite description which describes something unreal. Such a proposition as "x is unreal" only has meaning when "x" is a description, definite or indefinite; in that case the proposition will be true if "x" is a description which describes nothing. But whether the description "x" describes something or describes nothing, it is in any case not a constituent of the proposition in which it occurs; like "a unicorn" just now, it is not a subordinate group having a meaning of its own. All this results from the fact that, when "x" is a description, "x is unreal" or "x does not exist" is not nonsense, but is always significant and sometimes true.

We may now proceed to define generally the 171
meaning of propositions which contain ambiguous descriptions. Suppose we wish to make some statement about "a so-and-so," where "so-and-so's" are those objects that have a certain property ϕ, *i.e.* those objects x for which the propositional function ϕx is true. (*E.g.* if we take "a man" as our instance of "a so-and-so," ϕx will be "x is human.") Let us now wish to assert the property ψ of "a so-and-so," *i.e.* we wish to assert that "a so-and-so" has that property which x has when ψx is true. (*E.g.* in the case of "I met a man," ψx will be "I met x.") Now the proposition that "a so-and-so" has the property ψ is *not* a proposition of the form "ψx." If it were, "a so-and-so" would have to be identical with x for a suitable x; and although (in a sense) this may be true in some cases, it is certainly not true in such a case

as “a unicorn.” It is just this fact, that the statement that a so-and-so has the property ψ is not of the form ψx, which makes it possible for “a so-and-so” to be, in a certain clearly definable sense, “unreal.” The definition is as follows:—

> The statement that “an object having the property ϕ has the property ψ”

means:

> “The joint assertion of ϕx and ψx is not always false.”

So far as logic goes, this is the same proposition as might be expressed by “some ϕ’s are ψ’s”; but rhetorically there is a difference, because in the one case there is a suggestion of singularity, and in the other case of plurality. This, however, is not the important point. The important point is that, when rightly analysed, propositions verbally about “a so-and-so” are found to contain no constituent represented by this phrase. And that is why such propositions can be significant even when there is no such thing as a so-and-so.

The definition of *existence,* as applied to ambiguous descriptions, results from what was said at the end of the preceding chapter. We say that

"men exist" or "a man exists" if the | propositional 172
function "*x* is human" is sometimes true; and generally "a so-and-so" exists if "*x* is so-and-so" is sometimes true. We may put this in other language. The proposition "Socrates is a man" is no doubt *equivalent* to "Socrates is human," but it is not the very same proposition. The *is* of "Socrates is human" expresses the relation of subject and predicate; the *is* of "Socrates is a man" expresses identity. It is a disgrace to the human race that it has chosen to employ the same word "is" for these two entirely different ideas—a disgrace which a symbolic logical language of course remedies. The identity in "Socrates is a man" is identity between an object named (accepting "Socrates" as a name, subject to qualifications explained later) and an object ambiguously described. An object ambiguously described will "exist" when at least one such proposition is true, *i.e.* when there is at least one true proposition of the form "*x* is a so-and-so," where "*x*" is a name. It is characteristic of ambiguous (as opposed to definite) descriptions that there may be any number of true propositions of the above form—Socrates is a man, Plato is a man, etc. Thus "a man exists" follows from Socrates, or Plato, or anyone else. With definite descriptions, on the other hand, the corresponding

form of proposition, namely, "*x* is the so-and-so" (where "*x*" is a name), can only be true for one value of *x* at most. This brings us to the subject of definite descriptions, which are to be defined in a way analogous to that employed for ambiguous descriptions, but rather more complicated.

We come now to the main subject of the present chapter, namely, the definition of the word *the* (in the singular). One very important point about the definition of "a so-and-so" applies equally to "the so-and-so"; the definition to be sought is a definition of propositions in which this phrase occurs, not a definition of the phrase itself in isolation. In the case of "a so-and-so," this is fairly obvious: no one could suppose that "a man" was a definite object, which
could be defined by itself. | Socrates is a man, Plato 173
is a man, Aristotle is a man, but we cannot infer that "a man" means the same as "Socrates" means and also the same as "Plato" means and also the same as "Aristotle" means, since these three names have different meanings. Nevertheless, when we have enumerated all the men in the world, there is nothing left of which we can say, "This is a man, and not only so, but it is *the* 'a man,' the quintessential entity that is just an indefinite man without being anybody in particular." It is of course quite clear

that whatever there is in the world is definite: if it is a man it is one definite man and not any other. Thus there cannot be such an entity as "a man" to be found in the world, as opposed to specific men. And accordingly it is natural that we do not define "a man" itself, but only the propositions in which it occurs.

In the case of "the so-and-so" this is equally true, though at first sight less obvious. We may demonstrate that this must be the case, by a consideration of the difference between a *name* and a *definite description*. Take the proposition, "Scott is the author of *Waverley*." We have here a name, "Scott," and a description, "the author of *Waverley*," which are asserted to apply to the same person. The distinction between a name and all other symbols may be explained as follows:—

A name is a simple symbol whose meaning is something that can only occur as subject, *i.e.* something of the kind that, in Chapter XIII., we defined as an "individual" or a "particular." And a "simple" symbol is one which has no parts that are symbols. Thus "Scott" is a simple symbol, because, though it has parts (namely, separate letters), these parts are not symbols. On the other hand, "the author of *Waverley*" is not a simple symbol, because the sepa-

rate words that compose the phrase are parts which are symbols. If, as may be the case, whatever *seems* to be an "individual" is really capable of further analysis, we shall have to content ourselves with what may be called "relative individuals," which will be terms that, throughout the context in question, are never analysed and never occur | otherwise 174
than as subjects. And in that case we shall have correspondingly to content ourselves with "relative names." From the standpoint of our present problem, namely, the definition of descriptions, this problem, whether these are absolute names or only relative names, may be ignored, since it concerns different stages in the hierarchy of "types," whereas we have to compare such couples as "Scott" and "the author of *Waverley*," which both apply to the same object, and do not raise the problem of types. We may, therefore, for the moment, treat names as capable of being absolute; nothing that we shall have to say will depend upon this assumption, but the wording may be a little shortened by it.

We have, then, two things to compare: (1) a *name*, which is a simple symbol, directly designating an individual which is its meaning, and having this meaning in its own right, independently of the meanings of all other words; (2) a *description*, which consists of several words, whose meanings are already fixed, and from which results whatever is to be taken as the "meaning" of the description.

A proposition containing a description is not identical with what that proposition becomes when a name is substituted, even if the name names the same object as the description describes. "Scott is the author of *Waverley*" is obviously a different proposition from "Scott is Scott": the first is a fact in literary history, the second a trivial truism. And if we put anyone other than Scott in place of "the author of *Waverley*," our proposition would become false, and would therefore certainly no longer be the same proposition. But, it may be said, our proposition is essentially of the same form as (say) "Scott is Sir Walter," in which two names are said to apply to the same person. The reply is that, if "Scott is Sir Walter" really means "the person named 'Scott' is the person named 'Sir Walter,'" then the names are being used as descriptions: *i.e.* the individual, instead of being named, is being described as the person having that name. This is a way in which
names are frequently used | in practice, and there 175
will, as a rule, be nothing in the phraseology to show whether they are being used in this way or *as* names. When a name is used directly, merely to

indicate what we are speaking about, it is no part of the *fact* asserted, or of the falsehood if our assertion happens to be false: it is merely part of the symbolism by which we express our thought. What we want to express is something which might (for example) be translated into a foreign language; it is something for which the actual words are a vehicle, but of which they are no part. On the other hand, when we make a proposition about "the person called 'Scott,'" the actual name "Scott" enters into what we are asserting, and not merely into the language used in making the assertion. Our proposition will now be a different one if we substitute "the person called 'Sir Walter.'" But so long as we are using names *as* names, whether we say "Scott" or whether we say "Sir Walter" is as irrelevant to what we are asserting as whether we speak English or French. Thus so long as names are used *as* names, "Scott is Sir Walter" is the same trivial proposition as "Scott is Scott." This completes the proof that "Scott is the author of *Waverley*" is not the same proposition as results from substituting a name for "the author of *Waverley*," no matter what name may be substituted.

When we use a variable, and speak of a propositional function, ϕx say, the process of applying

general statements about ϕx to particular cases will consist in substituting a name for the letter "x," assuming that ϕ is a function which has individuals for its arguments. Suppose, for example, that ϕx is "always true"; let it be, say, the "law of identity," $x = x$. Then we may substitute for "x" any name we choose, and we shall obtain a true proposition. Assuming for the moment that "Socrates," "Plato," and "Aristotle" are names (a very rash assumption), we can infer from the law of identity that Socrates is Socrates, Plato is Plato, and Aristotle is Aristotle. But we shall commit a fallacy if we attempt to infer, without further premisses, that the author of *Waverley* is the author of *Waverley*. This results | from 176
what we have just proved, that, if we substitute a name for "the author of *Waverley*" in a proposition, the proposition we obtain is a different one. That is to say, applying the result to our present case: If "x" is a name, "$x = x$" is not the same proposition as "the author of *Waverley* is the author of *Waverley*," no matter what name "x" may be. Thus from the fact that all propositions of the form "$x = x$" are true we cannot infer, without more ado, that the author of *Waverley* is the author of *Waverley*. In fact, propositions of the form "the so-and-so is the so-and-so" are not always true: it is necessary

that the so-and-so should *exist* (a term which will be explained shortly). It is false that the present King of France is the present King of France, or that the round square is the round square. When we substitute a description for a name, propositional functions which are "always true" may become false, if the description describes nothing. There is no mystery in this as soon as we realise (what was proved in the preceding paragraph) that when we substitute a description the result is not a value of the propositional function in question.

We are now in a position to define propositions in which a definite description occurs. The only thing that distinguishes "the so-and-so" from "a so-and-so" is the implication of uniqueness. We cannot speak of "*the* inhabitant of London," because inhabiting London is an attribute which is not unique. We cannot speak about "the present King of France," because there is none; but we can speak about "the present King of England." Thus propositions about "the so-and-so" always imply the corresponding propositions about "a so-and-so," with the addendum that there is not more than one so-and-so. Such a proposition as "Scott is the author of *Waverley*" could not be true if *Waverley* had never been written, or if several people had

written it; and no more could any other proposition resulting from a propositional function ϕx by the substitution of "the author of *Waverley*" for "x." We may say that "the author of *Waverley*" means "the value of x for which 'x wrote | *Waverley*' is true." 177
Thus the proposition "the author of *Waverley* was Scotch," for example, involves:

(1) "x wrote *Waverley*" is not always false;
(2) "if x and y wrote *Waverley*, x and y are identical" is always true;
(3) "if x wrote *Waverley*, x was Scotch" is always true.

These three propositions, translated into ordinary language, state:

(1) at least one person wrote *Waverley*;
(2) at most one person wrote *Waverley*;
(3) whoever wrote *Waverley* was Scotch.

All these three are implied by "the author of *Waverley* was Scotch." Conversely, the three together (but no two of them) imply that the author of *Waverley* was Scotch. Hence the three together may be taken as defining what is meant by the proposition "the author of *Waverley* was Scotch."

We may somewhat simplify these three propositions. The first and second together are equivalent to: "There is a term c such that 'x wrote *Waverley*' is true when x is c and is false when x is not c." In other words, "There is a term c such that 'x wrote *Waverley*' is always equivalent to 'x is c.'" (Two propositions are "equivalent" when both are true or both are false.) We have here, to begin with, two functions of x, "x wrote *Waverley*" and "x is c," and we form a function of c by considering the equivalence of these two functions of x for all values of x; we then proceed to assert that the resulting function of c is "sometimes true," *i.e.* that it is true for at least one value of c. (It obviously cannot be true for more than one value of c.) These two conditions together are defined as giving the meaning of "the author of *Waverley* exists."

We may now define "the term satisfying the function ϕx exists." This is the general form of which the above is a particular case. "The author of *Waverley*" is "the term satisfying the function 'x wrote *Waverley*.'" And "the so-and-so" will always 178
involve reference to some propositional function, namely, that which defines the property that makes a thing a so-and-so. Our definition is as follows:—

"The term satisfying the function ϕx exists" means:

"There is a term c such that ϕx is always equivalent to 'x is c.'"

In order to define "the author of *Waverley* was Scotch," we have still to take account of the third of our three propositions, namely, "Whoever wrote *Waverley* was Scotch." This will be satisfied by merely adding that the c in question is to be Scotch. Thus "the author of *Waverley* was Scotch" is:

> "There is a term c such that (1) 'x wrote *Waverley*' is always equivalent to 'x is c,' (2) c is Scotch."

And generally: "the term satisfying ϕx satisfies ψx" is defined as meaning:

> "There is a term c such that (1) ϕx is always equivalent to 'x is c,' (2) ψc is true."

This is the definition of propositions in which descriptions occur.

It is possible to have much knowledge concerning a term described, *i.e.* to know many propositions concerning "the so-and-so," without actually knowing what the so-and-so is, *i.e.* without knowing any proposition of the form "x is the so-and-so," where "x" is a name. In a detective story propositions about "the man who did the deed" are accumulated, in the hope that ultimately they will suffice to demonstrate that it was A who did the deed. We may

even go so far as to say that, in all such knowledge as can be expressed in words—with the exception of "this" and "that" and a few other words of which the meaning varies on different occasions—no names, in the strict sense, occur, but what seem like names are really descriptions. We may inquire significantly whether Homer existed, which we could not do if "Homer" were a name. The proposition "the so-and-so exists" is significant, whether true or false; but if *a* is the so-and-so (where "*a*" is a name), the words "*a* exists" are meaningless. It is only of descriptions
—definite or indefinite—that existence can be sig- 179
nificantly asserted; for, if "*a*" is a name, it *must* name something: what does not name anything is not a name, and therefore, if intended to be a name, is a symbol devoid of meaning, whereas a description, like "the present King of France," does not become incapable of occurring significantly merely on the ground that it describes nothing, the reason being that it is a *complex* symbol, of which the meaning is derived from that of its constituent symbols. And so, when we ask whether Homer existed, we are using the word "Homer" as an abbreviated description: we may replace it by (say) "the author of the *Iliad* and the *Odyssey*." The same considerations apply to almost all uses of what look like proper

names.

When descriptions occur in propositions, it is necessary to distinguish what may be called "primary" and "secondary" occurrences. The abstract distinction is as follows. A description has a "primary" occurrence when the proposition in which it occurs results from substituting the description for "x" in some propositional function ϕx; a description has a "secondary" occurrence when the result of substituting the description for x in ϕx gives only *part* of the proposition concerned. An instance will make this clearer. Consider "the present King of France is bald." Here "the present King of France" has a primary occurrence, and the proposition is false. Every proposition in which a description which describes nothing has a primary occurrence is false. But now consider "the present King of France is not bald." This is ambiguous. If we are first to take "x is bald," then substitute "the present King of France" for "x," and then deny the result, the occurrence of "the present King of France" is secondary and our proposition is true; but if we are to take "x is not bald" and substitute "the present King of France" for "x," then "the present King of France" has a primary occurrence and the proposition is false. Confusion of primary and secondary occurrences is a ready source of fallacies where descriptions are concerned.

Descriptions occur in mathematics chiefly in the 180
form of *descriptive functions*, *i.e.* "the term having the relation R to y," or "the R of y" as we may say, on the analogy of "the father of y" and similar phrases. To say "the father of y is rich," for example, is to say that the following propositional function of c: "c is rich, and 'x begat y' is always equivalent to 'x is c,'" is "sometimes true," *i.e.* is true for at least one value of c. It obviously cannot be true for more than one value.

The theory of descriptions, briefly outlined in the present chapter, is of the utmost importance both in logic and in theory of knowledge. But for purposes of mathematics, the more philosophical parts of the theory are not essential, and have therefore been omitted in the above account, which has confined itself to the barest mathematical requisites.