Chapter 1

Basic Concepts of Set Theory

1.1 The concept of a set

A *set* is an abstract collection of distinct objects which are called the *members* or *elements* of that set. Objects of quite different nature can be members of a set, e.g. the set of red objects may contain cars, blood-cells, or painted representations. Members of a set may be concrete, like cars, blood-cells or physical sounds, or they may be abstractions of some sort, like the number two, or the English phoneme /p/, or a sentence of Chinese. In fact, we may arbitrarily collect objects into a set even though they share no property other than being a member of that set. The subject matter of set theory and hence of Part A of this book is what can be said about such sets disregarding the actual nature of their members.

Sets may be large, e.g. the set of human beings, or small, e.g. the set of authors of this book. Sets are either finite, e.g. the readers of this book or the set of natural numbers between 2 and 98407, or they are infinite, e.g. the set of sentences of a natural language or the set of natural numbers: zero, one, two, three, ... Since members of sets may be abstract objects, a set may in particular have another set as a member. A set can thus simultaneously *be* a member of another set and *have* other sets as members. This characteristic makes set theory a very powerful tool for mathematical and linguistic analysis.

A set may be a legitimate object even when our knowledge of its membership is uncertain or incomplete. The set of Roman Emperors is well-defined even though its membership is not widely known, and similarly the set of all former first-grade teachers is perfectly determined, although it may be hard to find out who belongs to it. For a set to be well-defined it must be clear *in principle* what makes an object qualify as a member of it. For our present purposes we may simply assume that, for instance, the set of red objects is well-defined, and disregard uncertainties about the exact boundary between red and orange or other sources of vagueness.

A set with only one member is called a *singleton*, e.g. the set consisting of you only, and there is one special set, the *empty* set or the *null* set, which has no members at all. The empty set may seem rather startling in the beginning, but it is the only possible representation of such things as the set of square circles or the set of all things not identical to themselves. Moreover, it is a mathematical convenience. If sets were restricted to having at least one member, many otherwise general statements about sets would have to contain a special condition for the empty set. As a matter of principle, mathematics strives for generality even when limiting or trivial cases must be included

We adopt the following set-theoretic notation: we write A, B, $C, \ldots$ for sets, and $a, b, c, \ldots$ or sometimes $x, y, z, \ldots$ for members of sets. The membership relation is written with a special symbol $\in$, so that $b \in A$ is read as 'b is a member of A'. It is convenient also to have a notation for the denial of the membership relation, written as $\notin$, so that $b \notin A$ is read as 'b is not a member of A'. Since sets may be members of other sets we will sometimes write $A \in B$, when the set A is a member of set B, disregarding the convention that members are written with lower case letters.

1.2 Specification of sets

There are three distinct ways to specify a set: (1) by listing all its members, (2) by stating a property which an object must have to qualify as a member of it, and (3) by defining a set of rules which generate its members. We discuss each method separately.

List notation: When a set is finite, its members can in principle be listed one by one until we have mentioned them all. To specify a set in list notation, the names of the members, written in a line and separated by commas, are enclosed in braces For example, the set whose members are the world's longest river, the first president of the United States and the number three could be written as

(1–1) {The Amazon River, George Washington, 3}

Several things must be noted here. First, in specifying a set, we use a name or some definite description of each of its members, but the set consists of the *objects named*, not of the names themselves. In our example, the first president of the United States, whose name happens to be 'George Washington', is a member of the set. But it is the man who belongs to the set, not his name Exactly the same set could have been described in the following way

(1–2) {The Amazon River, the first president of the United States, 3}

by using an alternative description for this individual. Of course, a set may also contain linguistic objects like names. To avoid confusion, names which are members of sets in their own right are put in single quotes. The set

(1–3) {The Amazon River, 'George Washington', 3}

should hence be distinguished from the set in (1-1), as it contains a river, a name and a number, but not the man who was the first president of the United States. It is important to realize that one and the same set may be described by several different lists, which *prima facie* have nothing in common except that they denote the same individuals.

Second, insofar as sets are concerned, it is an accidental feature of our left to right writing convention that the members are listed in a particular order. Contrary to what this notation may suggest, there is no first, second or third member in the set (1-1) A less misleading notation, which we sometimes use, is shown in (1-4) below; it avoids the suggestion of any ordering of its members (see the Venn diagrams in Sec. 6 below).

(1–4)

{ George Washington
3
The Amazon River }

The list notation is obviously more convenient to write and typeset, and is therefore usually preferred.

Another point about the list notation for sets is that writing the name of a member more than once does not change its membership status. Should we write

(1–5) $\{a, b, c, d, e, e, e, e\}$

we would have described exactly the same set as by writing

(1–6) $\{a, b, c, d, e\}$

This is a consequence of a fundamental principle of set theory: for a given object, either it is a member of a given set or it is not. There is no such thing as halfway, multiple or gradual membership in our set theory (although there have been attempts to construct theories of "fuzzy sets"; see Zadeh (1987)).

For large finite sets the list notation may be impractical and is abbreviated if some obvious pattern can be recognized in the list. For example, to list all multiples of five between zero and one hundred, we may write:

(1–7) $\{0, 5, 10, 15, \ldots, 95, 100\}$

Predicate notation: The list notation can be used, strictly speaking, only for finite sets, although it is sometimes used in elliptical form for well-known infinite sets such as the various sets of numbers. For example, the set of positive integers (whole numbers) is sometimes denoted by $\{1, 2, 3, 4, \ldots\}$. A better way to describe an infinite set is to indicate a property the members of the set share. The so-called predicate notation for this type of set description is illustrated by

(1–8) $\{x \mid x \text{ is an even number greater than } 3\}$

The vertical line following the first occurrence of the variable x is read 'such that'. The whole expression in (1-8) is read 'the set of all x such that x is an even number greater than 3.' Here x is a variable, which we may think of as an auxiliary symbol that stands for no particular object, but it indicates what the predicate is applied to. Note that the predicate notation describes finite and infinite sets in the same way (e.g., the predicate 'x is an even number between 3 and 9' specifies the finite set $\{4, 6, 8\}$) and that two predicates, if they are coextensive, will specify the same set. For example,

(1–9) $\{x \mid x \text{ is evenly divisible by 2 and is greater than or equal to } 4\}$

is the same set as (1-8).

A predicate may also define its members in relation to something else For instance, the set

(1–10) $\{x \mid x \text{ is a book and Mary owns } x\}$

contains the books that Mary owns.

Russell's Paradox: In the early years of set theory any conceivable property was thought to be a defining property of a set. But Bertrand Russell discovered in 1901 that a paradox could be obtained from an apparently acceptable set specification of that sort

Russell observed first that if sets are defined by properties of their members, some sets will turn out to be members of themselves and other sets will not. For example, the set of all elephants is not itself an elephant, and therefore is not a member of itself. But the set of all abstract concepts must contain itself as member, since a set is an abstract concept. The properties 'is a member of itself' and 'is not a member of itself' should therefore also be defining properties of sets. In particular, then, one could define a set U as the set of all those sets which are not members of themselves: $U = \{x \mid x \notin x\}$. Then we may ask of U whether it is a member of itself. Now two cases may obtain: (i) if U is *not* a member of itself, then it satisfies the defining characteristic of members of U, and therefore it must be a member of U, i.e., of itself; or (ii) if U *is* a member of itself, then it does not satisfy the defining property, hence it is *not* a member of U, i.e., of itself Since U either is or is not a member of U, the result is a logical paradox. The evident conclusion from this paradox is that there is no such set U, but nothing in Cantor's set theory excluded such a possible defining property. The discovery of the Russell paradox was therefore of great importance (many different but essentially equivalent versions have since been formulated), but it was all the more significant in light of the fact that logicians and mathematicians had been attempting to show that set theory could serve as a foundation for all of mathematics. The appearance of a paradox in the very foundations of set theory made some people doubtful of long-used and familiar mathematical notions, but mathematical practice continued as usual without being hampered by this foundational crisis. Many inventive and innovative solutions have been proposed to avoid the paradox, to resolve it or to make its consequences harmless. One such way, initially suggested by Russell, was *type theory*, which has found fruitful applications to natural language (e.g. in Montague Grammar; see Part D), as well as in the context of programming languages and their semantics, but it is beyond the scope of this book to discuss the type theories in general or any of the various other solutions to the set-theoretic paradoxes (see, however, the axiomatization of set theory in Chapter 8, section 2.8).

Recursive rules: Since finite sets specified simply by listing their members can never lead to such paradoxes, no changes had to be made for them. For infinite sets, the simplest way to avoid such paradoxes and still be able to define most sets of relevance to ordinary mathematics is to provide a rule for generating elements "recursively" from a finite basis. For example, the set $E = \{4, 6, 8, \ldots\}$ (=(1-8)=(1-9)) can be generated by the following rule:

(1–11) a) $4 \in E$
b) If $x \in E$, then $x + 2 \in E$
c) Nothing else belongs to E.

The first part of the rule specifies that 4 is a member of E; by applying the second part of the rule over and over, one ascertains that since $4 \in E$, then $6 \in E$; since $6 \in E$, then $8 \in E$; etc. The third part insures that no number is in E except in virtue of *a* and *b*

A rule for generating the members of a set has the following form: first, a finite number of members (often just one) are stated explicitly to belong to the set; then a finite number of if-then statements specifying some relation between members of the set are given, so that any member of the set can be reached by a chain of if-then statements starting from one of the members specified in the first part of the rule, and nothing that is not in the set can be reached by such a chain. We will consider such recursive devices in more detail in Chapter 8, section 1.1.

The earlier method of specifying a set by giving a defining property for its members has not been abandoned in practice, since it is often quite convenient and since paradoxical cases do not arise in the usual mathematical applications of set theory. Outside of specialized works on set theory itself, both methods are commonly used.

1.3 Set-theoretic identity and cardinality

We have already seen that different lists or different predicates may specify the same set. Implicitly we have assumed a notion of identity for sets which is one of the fundamental assumptions of set theory: two sets are identical if and only if they have exactly the same members. For instance, (1–12) $\{1, 2, 3, 4, 5, 6\}$

and

(1–13) $\{x \mid x \text{ is a positive integer less than } 7\}$

and

(1–14) a) $1 \in A$
b) if $x \in A$ and x is less than 6, then $x + 1 \in A$
c) nothing else is in A

are three different kinds of specifications, but because each picks out exactly the same members, we say that they specify *the same set*. We use the equals sign '=' for set-theoretic identity. Thus we may write, for example,

(1–15) $\{1, 2, 3, 4, 5, 6\} = \{x \mid x \text{ is a positive integer less than } 7\}$

The equals sign is also used in naming sets. For example, we might write 'let $B = \{1, 2, 3, 4, 5, 6\}$' to assign the name 'B' to the set in (1-12). The context will make it clear whether '=' is being used to stipulate the name of a set or to assert that two previously specified sets are identical.

A consequence of this notion of set-theoretic identity is that the empty set is unique, as its identity is fully determined by its absence of members. Thus the set of square circles and the set of non-self-identical things are the *same* set. Note that the empty list notation '$\{\}$' is never used for the empty set, but we have a special symbol '$\emptyset$' for it.

The number of members in a set A is called the *cardinality* of A, written $|A|$ or $\#(A)$. The cardinality of a finite set is given by one of the natural numbers. For example, the set defined in (1-12) has cardinality 6, and since (1-13) and (1-14) describe the same set, they describe sets of the same cardinality (of course distinct sets may also have the same cardinality). Infinite sets, too, have cardinalities, but they are not natural numbers. For example, the set of natural numbers itself has cardinality 'aleph-zero', written $\aleph_0$, which is not a natural number. We will take up the subject of infinite sets in more detail in Chapter 4

1.4 Subsets

When every member of a set A is also a member of a set B we call A a *subset* of B. We denote such a relation between sets by $A \subseteq B$. Note that

B may contain other members besides those of A, but this is not necessarily so. Thus the subset relation allows any set to be a subset of itself. If we want to exclude the case of a set being a subset of itself, the notion is called *proper subset*, and written as $A \subset B$. For the denial of the subset relation we put a slash across the subset symbol, e.g. $A \not\subseteq B$ means that A is not a subset of B, hence that A has at least one member which is not a member of B.

The following examples illustrate these concepts.

(1–16) a) $\{a, b, c\} \subseteq \{s, b, a, e, g, i, c\}$
b) $\{a, b, j\} \not\subseteq \{s, b, a, e, g, i, c\}$
c) $\{a, b, c\} \subset \{s, b, a, e, g, i, c\}$
d) $\emptyset \subset \{a\}$
e) $\{a, \{a\}\} \subseteq \{a, b, \{a\}\}$
f) $\{\{a\}\} \not\subseteq \{a\}$
g) $\{a\} \not\subseteq \{\{a\}\}$, but $\{a\} \in \{\{a\}\}$ (!!)

A curious consequence of the definition of subset is that the null set is a subset of every set. That is, for any set A whatever, $\emptyset \subseteq A$. Since $\emptyset$ has no members, the statement that *every* member of $\emptyset$ is also a member of A holds, even if vacuously. Alternatively, we could reason as follows. How could $\emptyset$ fail to be a subset of A? According to the definition of subset, there would have to be some member in $\emptyset$ that is not also a member of A. This is impossible since $\emptyset$ has no members at all, and so we cannot maintain that $\emptyset \not\subseteq A$. Since the argument does not depend in any way on what particular set is represented by A, it is true that $\emptyset \subseteq A$ for every A.

Note, however, that while $\emptyset \subseteq \{a\}$, for example $\{\emptyset\} \not\subseteq \{a\}$. The set $\{\emptyset\}$ has a member, namely $\emptyset$, and therefore is not the empty set. It is not true that every member of $\{\emptyset\}$ is also a member of $\{a\}$, so $\{\emptyset\} \not\subseteq \{a\}$.

Members of sets and *subsets* of sets both represent relationships of a part to a whole, but these relationships are quite different, and it is important not to confuse them. Subsets, as the name suggests, are *always sets*, whereas members may or may not be. Mars is a member of the set {Earth, Venus, Mars} but not a subset of it. The set containing Mars as its only member, {Mars}, *is* a subset of {Earth, Venus, Mars} because every member of the former is also a member of the latter. Further, whereas every set is a subset of itself, it is not clear whether a set can ever be a member of itself, as we saw above in the discussion of Russell's Paradox. Note how important it is here to distinguish between Mars, the planet, and {Mars}, the set.

Sets with sets as members provide the most opportunities for confusion. Consider, for example, the set $A = \{b, \{c\}\}$. The members of A are b and $\{c\}$. From the considerations in the preceding paragraph we see that $b \not\subseteq A$ and $\{b\} \subseteq A$. Similarly, $\{c\} \not\subseteq A$ because c is not a member of A, and $\{\{c\}\} \subseteq A$ because every member of $\{\{c\}\}$, namely, $\{c\}$, is a member of A. The reader should also verify the following statements concerning this example: $\{b\} \notin A$; $c \notin A$; $\{\{c\}\} \notin A$; $\{b, \{c\}\} \subseteq A$; $\{b, \{c\}\} \notin A$; $\{\{b, \{c\}\}\} \not\subseteq A$.

Another difference between subsets and members has to do with our previous remarks about sets of sets. We have seen that if $b \in X$ and $X \in \mathcal{C}$, it does not necessarily follow that $b \in \mathcal{C}$. The element b *could* be a member of $\mathcal{C}$, but if so this would be an accidental property of $\mathcal{C}$, not a necessary one. With inclusion, however, if $A \subseteq B$ and $B \subseteq C$, it is *necessarily true* that $A \subseteq C$; that is, if every member of A is also a member of B, and further if every member of B is also a member of C, then it must be true that every member of A is also a member of C. For example, $\{a\} \subseteq \{a, b\}$ and $\{a, b\} \subseteq \{a, b, c\}$ so it follows that $\{a\} \subseteq \{a, b, c\}$. On the other hand, $a \in \{a\}$ and $\{a\} \in \{\{a\}, b\}$, but $a \notin \{\{a\}, b\}$ (assuming of course that a and b are distinct).

1.5 Power sets

Sometimes we need to refer to the set whose members are all the subsets of a given set A. This set is called the power set of A, which we will write as $\wp(A)$. Suppose $A = \{a, b\}$; then the power set of A, $\wp(A)$, is $\{\{a\}, \{b\}, \{a, b\}, \emptyset\}$. The name 'power set' derives from the fact that if the cardinality of A is some natural number n, then $\wp(A)$ has cardinality 2^n, i.e., 2 raised to the n power, or $2 \times 2 \times 2 \times \ldots \times 2$ (n times). Sometimes the power set of A is denoted as 2^A.

1.6 Union and intersection

We now introduce two operations which take a pair of sets and produce another set.

The *union* of two sets A and B, written $A \cup B$, is the set where members are just the objects which are members of A or of B or of both. In the predicate notation the definition is

(1–17) $A \cup B =_{def} \{x \mid x \in A \text{ or } x \in B\}$

Note that the disjunction 'or' in (1-17) allows an object to be a member of both A and B. For this reason, the 'or' is an *inclusive* disjunction; (see Chapter 6, section 2). For example,

(1–18) Let $K = \{a, b\}$, $L = \{c, d\}$ and $M = \{b, d\}$, then

$$
\begin{array}{lclcl}
K \cup L & = & \{a, b, c, d\} & & \\
K \cup M & = & \{a, b, d\} & & \\
L \cup M & = & \{b, c, d\} & & \\
(K \cup L) \cup M & = & K \cup (L \cup M) & = & \{a, b, c, d\} \\
K \cup \emptyset & = & \{a, b\} & = & K \\
L \cup \emptyset & = & \{c, d\} & = & L
\end{array}
$$

Set-theoretic union can easily be generalized to apply to more than two sets, in which case we write the union sign in front of the set of sets to be operated on: e.g. $\bigcup\{K, L, M\}$ = the set of all elements in K or L or $M = \{a, b, c, d\}$. There is a nice method for visually representing set-theoretic operations, called *Venn diagrams*. Each set is drawn as a circle and its members are represented by points within it. The diagrams for two arbitrarily chosen sets are represented as partially intersecting – the most general case – as in Figure 1-1. The region designated '1' contains elements which are members of A but not of B; region 2, those things in B but not in A; and region 3, members of both B and A. Points in region 4 outside the diagram represent elements in neither set. Of course in particular instances one or more of these regions might turn out to be empty.

The Venn diagram for the union of A and B is then made by delineating all the regions contained in this set – shown in Figure 1-2 by shading areas 1, 2, and 3.

The second operation on arbitrary sets A and B produces a set whose members are just the members of *both* A and B. This operation is called the *intersection* of A and B, written as $A \cap B$. In predicate notation this operation would be defined as

(1–19) $A \cap B =_{def} \{x \mid x \in A \text{ and } x \in B\}$

For example, the intersection of the sets K and M of (1-18) is simply the singleton $\{b\}$, since b is the only object which is both a member of K and a member of M. Here are some more examples:

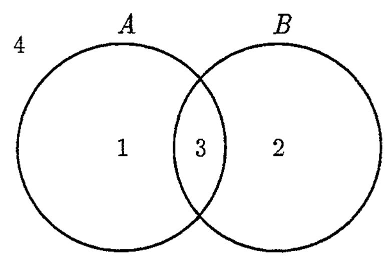

Figure 1–1: Venn diagram of two arbitrary sets A and B

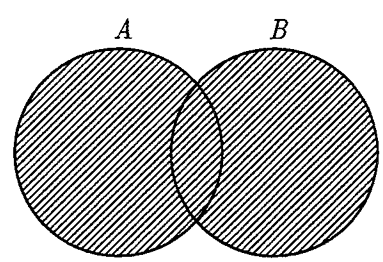

Figure 1–2: Set-theoretic union $A \cup B$.

$$
\text{(1–20)}\quad
\begin{array}{lclcl}
K \cap L & = & \emptyset & & \\
L \cap M & = & \{d\} & & \\
K \cap K & = & \{a, b\} & = & K \\
K \cap \emptyset & = & \emptyset & & \\
(K \cap L) \cap M & = & K \cap (L \cap M) & = & \emptyset \\
K \cap (L \cup M) & = & \{b\} & &
\end{array}
$$

The general case of intersection of arbitrary sets A and B is represented by the Venn diagram of Figure 1-3

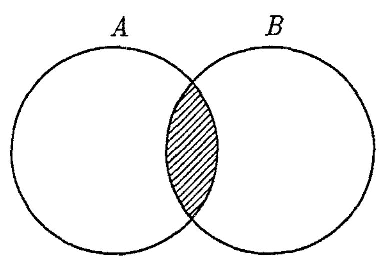

Figure 1–3: Set-theoretic intersection $A \cap B$.

Intersection may also be generalized to apply to three or more sets; e.g., $\bigcap\{K, L, M\} = \emptyset$. The intersection of three arbitrary sets A, B and C is shown in the Venn diagram of Figure 1-4. Here the black area represents what is common to the regions for $A \cap B$, $B \cap C$ and $A \cap C$. Obviously when more than three sets are involved, the Venn diagrams become very complex and of little use, but for simple cases they are a valuable visual aid in understanding set-theoretic concepts.

Problem: Construct a Venn diagram for the union of three arbitrary sets.

1.7 Difference and complement

Another binary operation on arbitrary sets A and B is the difference, written $A - B$, which 'subtracts' from A all objects which are in B. The predicate notation defines this operation as follows:

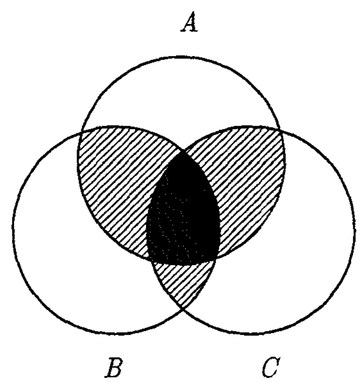

Figure 1–4: Venn diagram for $\bigcap\{A, B, C\}$ ($A \cap B$, $B \cap C$ and $A \cap C$ (shaded) and $\bigcap\{A, B, C\}$ (black)).

(1–21) $A - B =_{def} \{x \mid x \in A \text{ and } x \notin B\}$

$A - B$ is also called the *relative complement of* A and B. For instance for the particular sets L and M, given in (1-18), $L - M = \{c\}$, since c is the only member of L which is not a member of M. If A and B have no members in common, then nothing is taken from A; i.e., $A - B = A$. Note that although for all sets A, B: $A \cup B = B \cup A$ and $A \cap B = B \cap A$, it is not generally true that $A - B = B - A$. If one thinks of difference as a kind of subtraction, the fact that the order of the sets matters in this case is quite natural.

The Venn diagram for the set-theoretic difference $A - B$ is shown in Figure 1-5.

Some more examples:

(1–22)
$$\begin{aligned} K - M &= \{a\} \\ L - K &= \{c, d\} = L \\ M - L &= \{b\} \\ K - \emptyset &= \{a, b\} = K \\ \emptyset - K &= \emptyset \end{aligned}$$

This operation is to be distinguished from the *complement* of a set A,

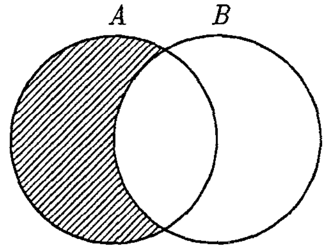

Figure 1–5: Set-theoretic difference $A - B$.

written A', which is the set consisting of everything not in A. In predicate notation

(1–23) $A' = \text{def } \{x \mid x \notin A\}$

Where do these objects come from which do not belong to A? The answer is that every statement involving sets is made against a background of assumed objects which comprise the *universe* (or *domain*) *of discourse* for that discussion. In talking about number theory, for example, the universe might be taken as the set of all positive and negative real numbers. A truly universal domain of discourse fixed once and for all, which would contain literally 'everything' out of which sets might be composed, is unfortunately impossible since it would contain paradoxical objects such as 'the set of all sets'. Therefore, the universe of discourse varies with the discussion, much as the interpretation of the English words 'everything' and 'everyone' tends to be implicitly restricted by the context of discourse When no other specified name has been given to the universe of discourse in a particular discussion, we conventionally use the symbol U for it. When it is clear from the context or irrelevant to the discussion at hand, the universe of discourse may not be explicitly mentioned at all, but the operation of complement is not well-defined without it. The complement of a set A, then, is the set of all objects in the universe of discourse which are not in A, i.e.,

(1–24) $A' = U - A$

We see that in (1-23) the variable x in the predicate notation is implicitly understood to range over (i e , take its values from) the set- theoretic universe U (and the same is true, incidentally, in (1-17) and (1-19)).

The Venn diagram with a shaded section for the complement of A is shown in Figure 1-6.

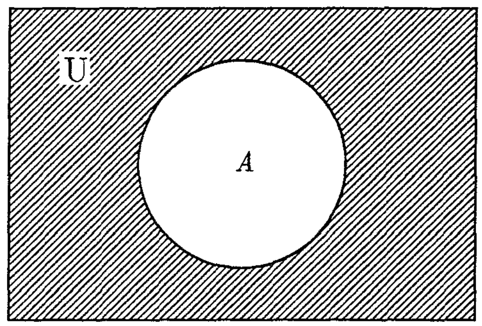

Figure 1–6: The set-theoretic complement A'.

1.8 Set-theoretic equalities

There are a number of general laws pertaining to sets which follow from the foregoing definitions of union, intersection, subset, etc. A useful selection of these is shown in Figure 1-7, where they are grouped (generally in pairs – one for union, one for intersection) under their more or less traditional names. We are not yet in a position to offer formal proofs that these statements really do hold for any arbitrarily chosen sets X, Y, and Z (we will take this up in Chapter 7, section 6), but for now we may perhaps try to convince ourselves of their truth by reflecting on the relevant definitions or constructing some Venn diagrams.

It is easy to see that for any set X, $X \cup X$ is the same as X, since everything which is in X *or* in X simply amounts to everything which is in X. And similarly for everything which is in X *and* in X, so $X \cap X = X$.

Likewise, everything which is in X or in Y (or both) is the same as everything which is in Y or in X (or both); thus, $X \cup Y = Y \cup X$. The argument for intersection is similar.

1. *Idempotent Laws*
   (a) $X \cup X = X$ (b) $X \cap X = X$
2. *Commutative Laws*
   (a) $X \cup Y = Y \cup X$ (b) $X \cap Y = Y \cap X$
3. *Associative Laws*
   (a) $(X \cup Y) \cup Z = X \cup (Y \cup Z)$ (b) $(X \cap Y) \cap Z = X \cap (Y \cap Z$
4. *Distributive Laws*
   (a) $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$
   (b) $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$
5. *Identity Laws*
   (a) $X \cup \emptyset = X$ (c) $X \cap \emptyset = \emptyset$
   (b) $X \cup U = U$ (d) $X \cap U = X$
6. *Complement Laws*
   (a) $X \cup X' = U$ (c) $X \cap X' = \emptyset$
   (b) $(X')' = X$ (d) $X - Y = X \cap Y'$
7. *DeMorgan's Law*
   (a) $(X \cup Y)' = X' \cap Y'$ (b) $(X \cap Y)' = X' \cup Y'$
8. *Consistency Principle*
   (a) $X \subseteq Y$ iff $X \cup Y = Y$ (b) $X \subseteq Y$ iff $X \cap Y = X$

Figure 1–7: Some fundamental set-theoretic equalities.

The Associative Laws state that the order in which we combine three sets by the operation of union does not matter, and the same is true if the operation is intersection To see that these hold, imagine the construction of the appropriate Venn diagrams. We have three intersecting circles labelled X, Y, and Z. We shade $X \cup Y$ first and then shade Z. The result is shading of the entire area inside the three circles, and this corresponds to $(X \cup Y) \cup Z$. Now we start over and shade $Y \cup Z$ first and then X. The result is the same.

The construction of the Venn diagrams to illustrate the Distributive Laws is somewhat trickier. In Figure 1-8 we show a Venn diagram for $X \cap (Y \cup Z)$. To make it more perspicuous, X has been shaded with vertical lines and $Y \cup Z$ horizontally. The intersection of these two sets is then represented by the cross-hatched area. Figure 1-9 shows the corresponding diagram for $(X \cap Y) \cup (X \cap Z)$ $X \cap Y$ is shaded vertically and $X \cap Z$ horizontally; thus, the union is represented by the area shaded in either (or both) directions. The reader should now be able to construct the Venn diagram for case (a) of the Distributive Laws.

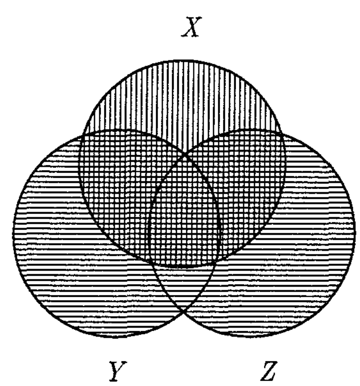

Figure 1–8: Venn diagram for $X \cap (Y \cup Z)$
(X shaded vertically, $Y \cup Z$ shaded
horizontally, $X \cap (Y \cup Z)$ cross-hatched).

The Identity Laws are evident from the definitions of union, intersection, the null set, and the universal set. Everything which is in X or in $\emptyset$ just amounts to everything which is in X, etc. The Complement Laws are likewise easily grasped from the definitions of complement with perhaps a look at

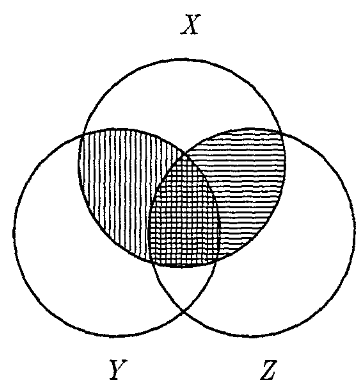

Figure 1–9: Venn diagram for
$(X \cap Y) \cup (X \cap Z)$ ($X \cap Y$ shaded vertically,
$X \cap Z$ shaded horizontally,
$(X \cap Y) \cup (X \cap Z)$ the entire shaded area).

the Venn diagram in Figure 1-6. Case (d) becomes less baffling if we look at Figure 1-5 and consider the area corresponding to the intersection of A with the complement of B.

DeMorgan's Laws are a symmetrical pair. Case (a): everything which is in neither X nor Y is the same as everything which is not in X and not in Y. Case (b): everything which is not in both X and Y is either not in X or not in Y (or in neither). This case is less immediately evident, and a Venn diagram will help.

The Consistency Principle is so called because it is concerned with the mutual consistency of the definitions of union, intersection, and subset. If we think of a Venn diagram in which the circle for X lies entirely inside the circle for Y (representing $X \subseteq Y$), then it is easy to see that $X \cup Y = Y$. On the other hand, if we know that $X \cup Y = Y$, then in the standard Venn diagram the region corresponding to elements which are in X but not in Y must be empty (otherwise, the union would not be equal to Y). Thus, X's members lie entirely in the Y circle; so $X \subseteq Y$. The (b) case is similar.

It may help in getting a grasp on some of these laws if one considers analogues from algebra. The operation of $+$ (addition) and $*$ (multiplication) obey a commutative law:

(1–25) for all numbers x, y, $x + y = y + x$ and $x * y = y * x$

and an associative law:

(1–26) for all numbers x, y, z, $(x + y) + z = x + (y + z)$ and $(x * y) * z = x * (y * z)$

but neither is idempotent: in general it is not true that $x + x = x$ nor that $x * x = x$ However, there is a distributive law relating $*$ and $+$ as follows:

(1–27) for all numbers x, y, z, $x * (y + z) = (x * y) + (x * z)$

but no such law holds if $*$ and $+$ are interchanged; i.e., it is not in general true that $x + (y * z) = (x + y) * (x + z)$. (For example, let $x = 1$, $y = 2$, and $z = 3$; then the left side is 7 and the right side is 12)

Arithmetic analogues of the Identity Laws are $x + 0 = x$, $x * 0 = 0$, and $x * 1 = x$ with 0 playing the role of the null set and 1 that of the universal set. (But this analogy, too, breaks down: $x + 1$ does not equal 1.)

What we have seen then is that there is an algebra of sets which is in some respects analogous to the familiar algebra involving addition and multiplication but which has its own peculiar properties as well. We will encounter this structure once more when we take up the logic of statements in Chapter 6, and we will discover in Chapter 12 that both are instances of what is called a Boolean algebra.

For the moment our concern is not with the structure of this algebra but rather to show how these equalities can be used in the manipulation of set-theoretic expressions. The idea is that in any set-theoretic expression a set may always be replaced by one equal to it. The result will then be an expression which denotes the same set as the original expression. For example, in $A \cap (B \cup C)'$ we may replace $(B \cup C)'$ by its equivalent, $B' \cap C'$ (citing DeMorgan's Laws), to obtain $A \cap (B' \cap C')$. Since $(B \cup C)'$ and $B' \cap C'$ have the same members, so do $A \cap (B \cup C)'$ and $A \cap (B' \cap C')$.

This technique can be used to simplify a complex set-theoretic expression, as in (1-28) below, or to demonstrate the truth of other statements about sets, as in (1-29) and (1-30). It is usually convenient to arrange such demonstrations as a vertical sequence in which each line is justified by reference to the law employed in deriving it from the preceding line.

(1–28) *Example:* Simplify the expression $(A \cup B) \cup (B \cap C)'$

1.	$(A \cup B) \cup (B \cap C)'$	
2.	$(A \cup B) \cup (B' \cup C')$	DeM.
3.	$A \cup (B \cup (B' \cup C'))$	Assoc
4.	$A \cup ((B \cup B') \cup C')$	Assoc.
5.	$A \cup (U \cup C')$	Compl.
6.	$A \cup (C' \cup U)$	Comm.
7.	$A \cup U$	Ident.
8.	U	Ident.

(1–29) *Example:* Show that $(A \cap B) \cap (A \cap C)' = A \cap (B - C)$.

1.	$(A \cap B) \cap (A \cap C)'$	
2.	$(A \cap B) \cap (A' \cup C')$	DeM
3.	$A \cap (B \cap (A' \cup C'))$	Assoc.
4.	$A \cap ((B \cap A') \cup (B \cap C'))$	Distr.
5.	$(A \cap (B \cap A')) \cup (A \cap (B \cap C'))$	Distr.
6.	$(A \cap (A' \cap B)) \cup (A \cap (B \cap C'))$	Comm.
7.	$((A \cap A') \cap B) \cup (A \cap (B \cap C'))$	Assoc.
8.	$(\emptyset \cap B) \cup (A \cap (B \cap C'))$	Compl.
9.	$(B \cap \emptyset) \cup (A \cap (B \cap C'))$	Comm.
10.	$\emptyset \cup (A \cap (B \cap C'))$	Ident.
11.	$(A \cap (B \cap C')) \cup \emptyset$	Comm.
12.	$A \cap (B \cap C')$	Ident.
13.	$A \cap (B - C)$	Compl

(1–30) *Example:* Show that $X \cap Y \subseteq X \cup Y$.

By the Consistency Principle this expression is true iff $(X \cap Y) \cap (X \cup Y) = X \cap Y$. We demonstrate the latter.

1.	$(X \cap Y) \cap (X \cup Y)$	
2.	$((X \cap Y) \cap X) \cup ((X \cap Y) \cap Y)$	Distr.
3.	$(X \cap (X \cap Y)) \cup ((X \cap Y) \cap Y)$	Comm.
4.	$((X \cap X) \cap Y) \cup ((X \cap Y) \cap Y)$	Assoc.
5.	$((X \cap X) \cap Y) \cup (X \cap (Y \cap Y))$	Assoc.
6.	$(X \cap Y) \cup (X \cap Y)$	Idemp. (twice)
7.	$X \cap Y$	Idemp.

Such arrays constitute formal proofs (of the fact that, in each of these cases, the set in the last line is equal to that in the first line) We will take up the topic of proofs in due course, but the reader who attempts such derivations in the exercises will no doubt encounter a notoriously troublesome problem connected with proofs; namely, while it is relatively simple to verify that a given proof is correct, it may be very difficult to find the one one wants. So if presented with a problem such as (1-29), one might have to try many unsuccessful paths before finding one that leads to the desired final expression. A certain amount of cutting and trying is therefore to be expected

Exercises

1. Given the following sets:

$$A = \{a, b, c, 2, 3, 4\} \quad E = \{a, b, \{c\}\}$$
$$B = \{a, b\} \quad F = \emptyset$$
$$C = \{c, 2\} \quad G = \{\{a, b\}, \{c, 2\}\}$$
$$D = \{b, c\}$$

classify each of the following statements as true or false

(a)	$c \in A$	(g)	$D \subset A$	(m)	$B \subseteq G$
(b)	$c \in F$	(h)	$A \subseteq C$	(n)	$\{B\} \subseteq G$
(c)	$c \in E$	(i)	$D \subseteq E$	(o)	$D \subseteq G$
(d)	$\{c\} \in E$	(j)	$F \subseteq A$	(p)	$\{D\} \subseteq G$
(e)	$\{c\} \in C$	(k)	$E \subseteq F$	(q)	$G \subseteq A$
(f)	$B \subseteq A$	(l)	$B \in G$	(r)	$\{\{c\}\} \subseteq E$

2. For any arbitrary set S,

   (a) is S a member of $\{S\}$?

   (b) is $\{S\}$ a member of $\{S\}$?

   (c) is $\{S\}$ a subset of $\{S\}$?

   (d) what is the set whose only member is $\{S\}$?

3. Write a specification by rules and one by predicates for each of the following sets. Remember that there is no order assumed in the list, so you cannot use words like 'the first' or 'the latter'. Recall also that a recursive rule may contain more than one if-then statement.

   (a) $\{5, 10, 15, 20, \ldots\}$

   (b) $\{7, 17, 27, 37, \ldots\}$

   (c) $\{300, 301, 302, \ldots, 399, 400\}$

   (d) $\{3, 4, 7, 8, 11, 12, 15, 16, 19, 20, \ldots\}$

(e) $\{0, 2, -2, 4, -4, 6, -6, \ldots\}$

(f) $\{1, 1/2, 1/4, 1/8, 1/16, \ldots\}$

4. Consider the following sets:

$$\begin{array}{lll} S1 = \{\{\emptyset\}, \{A\}, A\} & & S6 = \emptyset \\ S2 = A & & S7 = \{\emptyset\} \\ S3 = \{A\} & & S8 = \{\{\emptyset\}\} \\ S4 = \{\{A\}\} & & S9 = \{\emptyset, \{\emptyset\}\} \\ S5 = \{\{A\}, A\} & & \end{array}$$

Answer the following questions Remember that the members of a set are the items separated by commas, if there is more than one, between the outermost braces only; a subset is formed by enclosing within braces zero or more of the members of a given set, separated by commas.

(a) Of the sets $S1$ - $S9$ which are members of $S1$?

(b) which are subsets of $S1$?

(c) which are members of $S9$?

(d) which are subsets of $S9$?

(e) which are members of $S4$?

(f) which are subsets of $S4$?

5. Specify each of the following sets by listing its members:

(a) $\wp\{a, b, c\}$ (d) $\wp\{\emptyset\}$

(b) $\wp\{a\}$ (e) $\wp\wp\{a, b\}$

(c) $\wp\emptyset$

6. Given the sets $A, \ldots, G$ as in Exercise 1, list the members of each of the following:

(a) $B \cup C$ (g) $A \cap E$ (m) $B - A$

(b) $A \cup B$ (h) $C \cap D$ (n) $C - D$

(c) $D \cup E$ (i) $B \cap F$ (o) $E - F$

(d) $B \cup G$ (j) $C \cap E$ (p) $F - A$

(e) $D \cup F$ (k) $B \cap G$ (q) $G - B$

(f) $A \cap B$ (l) $A - B$

7. Given the sets in Exercise 1, assume that the universe of discourse is $\bigcup\{A, B, C, D, E, F, G\}$. List the members of the following sets:

(a)	$(A \cap B) \cup C$	(h)	$D' \cap E'$
(b)	$A \cap (B \cup C)$	(i)	$F \cap (A - B)$
(c)	$(B \cup C) - (C \cup D)$	(j)	$(A \cap B) \cup U$
(d)	$A \cap (C - D)$	(k)	$(C \cup D) \cap U$
(e)	$(A \cap C) - (A \cap D)$	(l)	$C \cap D'$
(f)	G'	(m)	$G \cup F'$
(g)	$(D \cup E)'$	(n)	$(B \cap C)'$

8. Let $A = \{a, b, c\}$, $B = \{c, d\}$ and $C = \{d, e, f\}$.

   (a) What are:

(i)	$A \cup B$	(v)	$B \cup \emptyset$
(ii)	$A \cap B$	(vi)	$A \cap (B \cap C)$
(iii)	$A \cup (B \cap C)$	(vii)	$A - B$
(iv)	$C \cup A$		

   (b) Is a a member of $\{A, B\}$?

   (c) Is a a member of $A \cup B$?

9. Show by using the set-theoretic equalities in Figure 1-7 for any sets A, B, and C,

   (a) $((A \cup C) \cap (B \cup C')) \subseteq (A \cup B)$

   (b) $A \cap (B - A) = \emptyset$

10. Show that the Distributive Law 4(a) is true by constructing Venn diagrams for $X \cup (Y \cap Z)$ and $(X \cup Y) \cap (X \cup Z)$.

11. The *symmetric difference* of two sets A and B, denoted $A + B$, is defined as the set whose members are in A or in B but not in both A and B, i.e.

$$A + B =_{def} (A \cup B) - (A \cap B)$$

   (a) Draw the Venn diagram for the symmetric difference of two sets.

   (b) Show that $A + B = (A - B) \cup (B - A)$ by means of the set-theoretic equalities in Figure 1-7. Verify that the Venn diagram for $(A - B) \cup (B - A)$ is equivalent to that in (a).

   (c) Show that for all sets A and B, $A + B = B + A$.

(d) Express each of the following in terms of union, intersection, and complementation, and simplify using the set-theoretic equalities.

(i) $A + A$ (iv) $A + B$, where $A \subseteq B$
(ii) $A + U$ (v) $A + B$, where $A \cap B = \emptyset$
(iii) $A + \emptyset$

(e) Show that $((A - B) + (B - A)) = A + B$

(f) Show that $(A + B) \subseteq B$ iff $A \subseteq B$

12. Call adjectives which are correctly predicated of themselves 'autological' and those which are not, 'heterological.' For example, 'English' and 'short' are autological, but 'French' and 'long' are heterologial. Show that when we ask whether the adjective 'heterological' is heterological or autological we are led to a contradiction like that in Russell's Paradox. This is known as Grelling's Paradox.

Chapter 2

Relations and Functions

2.1 Ordered pairs and Cartesian products

Recall that there is no order imposed on the members of a set. We can, however, use ordinary sets to define an *ordered pair*, written $\langle a, b \rangle$ for example, in which a is considered the *first member* and b is the *second member* of the pair. The definition is as follows:

(2–1) $\langle a, b \rangle =_{def} \{\{a\}, \{a, b\}\}$

The first member of $\langle a, b \rangle$ is taken to be the element which occurs in the singleton $\{a\}$, and the second member is the one which is a member of the other set $\{a, b\}$, but not of $\{a\}$. Now we have the necessary properties of an ordering since in general $\langle a, b \rangle \neq \langle b, a \rangle$. This is so because we have $\{\{a\}, \{a, b\}\} = \{\{b\}, \{a, b\}\}$ (that is, $\langle a, b \rangle = \langle b, a \rangle$), if and only if we have $a = b$. Of course, this definition can be extended to ordered triples and in general ordered n-tuples for any natural number n. Ordered triples are defined as

(2–2) $\langle a, b, c \rangle =_{def} \langle \langle a, b \rangle, c \rangle$

It might have been intuitively simpler to start with ordered sets as an additional primitive, but mathematicians like to keep the number of primitive notions to a minimum.

If we have two sets A and B, we can form ordered pairs from them by taking an element of A as the first member of the pair and an element of B

as the second member. The *Cartesian product* of A and B, written $A \times B$, is the set consisting of all such pairs. The predicate notation defines it as

(2–3) $A \times B =_{def} \{\langle x, y \rangle \mid x \in A \text{ and } y \in B\}$

Note that according to the definition if either A or B is $\emptyset$, then $A \times B = \emptyset$. Here are some examples of Cartesian products:

(2–4) Let $K = \{a, b, c\}$ and $L = \{1, 2\}$, then

$$\begin{aligned} K \times L &= \{\langle a,1 \rangle, \langle a,2 \rangle, \langle b,1 \rangle, \langle b,2 \rangle, \langle c,1 \rangle, \langle c,2 \rangle\} \\ L \times K &= \{\langle 1,a \rangle, \langle 2,a \rangle, \langle 1,b \rangle, \langle 2,b \rangle, \langle 1,c \rangle, \langle 2,c \rangle\} \\ L \times L &= \{\langle 1,1 \rangle, \langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle\} \end{aligned}$$

It is important to remember that the members of a Cartesian product are *not* ordered with respect to each other. Although each member is an ordered pair, the Cartesian product is itself an unordered set of them.

Given a set M of ordered pairs it is sometimes of interest to determine the smallest Cartesian product of which M is a subset. The smallest A and B such that $M \subseteq A \times B$ can be found by taking $A = \{a \mid \langle a, b \rangle \in M$ for some $b\}$ and $B = \{b \mid \langle a, b \rangle \in M$ for some $a\}$. These two sets are called the *projections of M onto the first and the second coordinates*, respectively. For example, if $M = \{\langle 1,1 \rangle, \langle 1,2 \rangle, \langle 3,2 \rangle\}$, the set $\{1,3\}$ is the projection onto the first coordinate, and $\{1,2\}$ the projection onto the second coordinate. Thus $\{1,3\} \times \{1,2\}$ is the smallest Cartesian product of which M is a subset.

2.2 Relations

We have a natural understanding of relations as the sort of things that hold or do not hold between objects. The relation 'mother of' holds between any mother and her children but not between the children themselves, for instance. Transitive verbs often denote relations; e.g., the verb 'kiss' can be regarded as denoting an abstract relation between pairs of objects such that the first kisses the second. The subset relation was defined above as a relation between sets. Objects in a set may be related to objects in the same or another set. We write Rab or equivalently aRb if the relation R holds between objects a and b. We also write $R \subseteq A \times B$ for a relation between objects from two sets A and B, which we call a relation *from A to*

B. A relation holding of objects from a single set A is called a relation *in* A. The projection of R onto the first coordinate is called the *domain* of R and the projection of R onto the second coordinate is called the *range* of R. A relation R from A to B thus can be viewed as a subset of the Cartesian product $A \times B$. (There are unfortunately no generally accepted terms for the sets A and B of which the domain and the range are subsets.) It is important to realize that this is a *set-theoretic* reduction of the relation R to a set of ordered pairs, i.e. $\{\langle a, b \rangle \mid aRb\}$. For example, the relation 'mother of' defined on the set H of all human beings would be a set of ordered pairs in $H \times H$ such that in each pair the first member is mother of the second member. We may visually represent a relation R between two sets A and B by arrows in a diagram displaying the members of both sets.

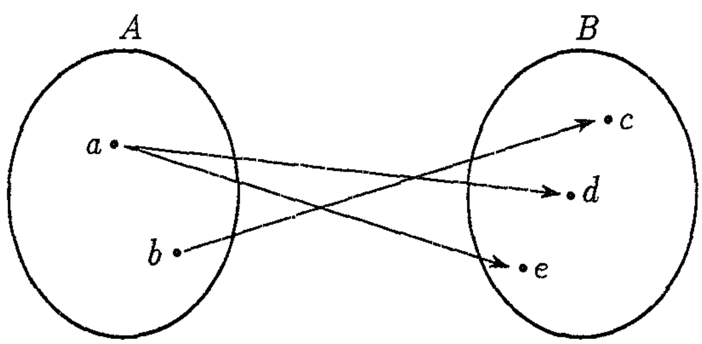

Figure 2–1: Relation R: $A \rightarrow B$.

In Figure 2-1, $A = \{a, b\}$ and $B = \{c, d, e\}$ and the arrows represent a set-theoretic relation $R = \{\langle a, d\rangle, \langle a, e \rangle, \langle b, c \rangle\}$. Note that a relation may relate one object in its domain to more than one object in its range. The complement of a relation $R \subseteq A \times B$, written R', is set-theoretically defined as

(2–5) $R' =_{def} (A \times B) - R$

Thus R' contains all ordered pairs of the Cartesian product which are not members of the relation R. Note that $(R')' = R$. The *inverse* of a relation $R \subseteq A \times B$, written R^{-1}, has as its members all the ordered pairs in R, with their first and second elements reversed. For example, let $A = \{1, 2, 3\}$ and let $R \subseteq A \times A$ be $\{\langle 3, 2 \rangle, \langle 3, 1 \rangle, \langle 2, 1 \rangle\}$, which is the 'greater than' relation in A. The complement relation R' is $\{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 3 \rangle\}$, the 'less than or equal to' relation in A. The inverse of R, R^{-1}, is $\{\langle 2,3 \rangle, \langle 1, 3\rangle, \langle 1,2 \rangle\}$, the 'less than' relation in A. Note that $(R^{-1})^{-1} = R$, and that if $R \subseteq A \times B$, then $R^{-1} \subseteq B \times A$, but $R' \subseteq A \times B$.

We have focused in this discussion on *binary* relations, i.e., sets of ordered pairs, but analogous remarks could be made about relations which are composed of ordered triples, quadruples, etc., i.e., *ternary*, *quaternary*, or just n-place relations.

2.3 Functions

A function is generally represented in set-theoretic terms as a special kind of relation. A relation R from A to B is a function if and only if it meets both of the following conditions:

1. Each element in the domain is paired with just one element in the range.
2. The domain of R is equal to A.

This amounts to saying that a subset of a Cartesian product $A \times B$ can be called a function just in case every member of A occurs exactly once as a first coordinate in the ordered pairs of the set.

As an example, consider the sets $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. The following relations from A to B are functions:

(2–6)
$$\begin{aligned} P &= \{\langle a,1\rangle, \langle b,2\rangle, \langle c,3\rangle\} \\ Q &= \{\langle a,3\rangle, \langle b,4\rangle, \langle c,1\rangle\} \\ R &= \{\langle a,3\rangle, \langle b,2\rangle, \langle c,2\rangle\} \end{aligned}$$

The following relations from A to B are not functions:

(2–7)
$$\begin{aligned} S &= \{\langle a,1\rangle, \langle b,2\rangle\} \\ T &= \{\langle a,2\rangle, \langle b,3\rangle, \langle a,3\rangle, \langle c,1\rangle\} \\ V &= \{\langle a,2\rangle, \langle a,3\rangle, \langle b,4\rangle\} \end{aligned}$$

S fails to meet condition 2 because the set of first members, namely $\{a, b\}$, is not equal to A. T does not satisfy condition 1, since a is paired with both 2 and 3. In relation V both conditions are violated.

Much of the terminology used in talking about functions is the same as that for relations. We say that a function that is a subset of $A \times B$ is a function *from A to B*, while one in $A \times A$ is said to be a function *in A*. The notation '$F: A \to B$' is used for 'F is a function from A to B.' Elements in the domain of a function are sometimes called *arguments* and their correspondents in the range, *values*. Of function P in (2-6), for example, one may say that it takes on the value 3 at argument c. The usual way to denote this fact is $P(c) = 3$, with the name of the function preceding the argument, which is enclosed in parentheses, and the corresponding value to the right of the equal sign.

'Transformation,' 'map,' 'mapping,' and 'correspondence' are commonly used synonyms for 'function,' and often '$F(a) = 2$' is read as 'F maps a into 2.' Such a statement gives a function the appearance of an active process that changes arguments into values. This view of functions is reinforced by the fact that in most of the functions commonly encountered in mathematics the pairing of arguments and values can be specified by a formula containing operations such as addition, multiplication, division, etc. For example, $F(x) = 2x + 1$ is a function which, when defined on the set of integers, pairs 1 with 3, 2 with 5, 3 with 7, and so on. This can be thought of as a rule which says, "To find the value of F at x, multiply x by 2 and add 1." Later in this book it may prove to be necessary to think of functions as dynamic processes transforming objects as their input into other objects as their output, but for the present, we adhere to the more static set-theoretic perspective. Thus, the function $F(x) = 2x + 1$ will be regarded as a set of ordered pairs which could be defined in predicate notation as

(2–8) $F = \{\langle x, y\rangle \mid y = 2x + 1\}$ (where x and y are integers)

Authors who regard functions as processes sometimes refer to the set of ordered pairs obtained by applying the process at each element of the domain as the *graph* of the function. The connection between this use of "graph" and a representation consisting of a line drawn in a coordinate system is not accidental.

We should also note that relations which satisfy condition 1 above but perhaps fail condition 2 are sometimes regarded as functions, but if so, they are customarily designated as 'partial functions.' For example, the function which maps an ordered pair of real numbers $\langle a, b\rangle$ into the quotient of a divided by b (e.g., it maps $\langle 6, 2\rangle$ into 3 and $\langle 5, 2\rangle$ into 2.5) is not defined when $b = 0$. But it is single-valued – each pair for which it is defined is associated with a unique value – and thus it meets condition 1. Strictly speaking, by our definition it is not a function, but it could be called a partial function. A partial function is thus a total function on some subset of the domain. Henceforth, we will use the term 'function,' if required, to indicate a single-valued mapping whose domain may be less than the set A containing the domain.

It is sometimes useful to state specifically whether or not the range of a function from A to B is equal to the set B. Functions from A to B in general are said to be *into* B If the range of the function equals B, however, then the function is *onto* B. (Thus *onto* functions are also *into*, but not necessarily conversely.) In Figure 2-2 three functions are indicated by the same sort of diagrams we introduced previously for relations generally It should be apparent that functions F and G are *onto* but H is not. All are of course *into*.

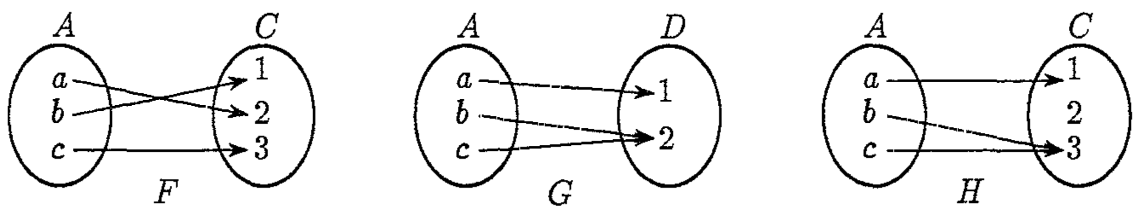

Figure 2–2: Illustration of onto and into functions.

A function $F: A \to B$ is called a *one-to-one* function just in case no member of B is assigned to more than one member of A. Function F in Figure 2-2 is one-to-one, but G is not (since both b and c are mapped into 2), nor is H (since $H(b) = H(c) = 3$). The function F defined in (2-8) is one-to-one since for each odd integer y there is a unique integer x such that $y = 2x + 1$. Note that F is not onto the set of integers since no even integer is the value of F for any argument x. Functions which are not necessarily one-to-one may be termed *many to one*. Thus all functions are many-to-one strictly speaking, and some but not all of them are one-to-one. It is usual to apply the term "many-to-one", however, only to those functions which are not in fact one-to-one.

A function which is both one-to-one and onto (F in Figure 2-2 is an example) is called a *one-to-one correspondence* Such functions are of special interest because their inverses are also functions (Note that the definitions of the inverse and the complement of a relation apply to functions as well) The inverse of G in Figure 2-2 is not a function since 2 is mapped into both b and c, and in H^{-1} the element 2 has no correspondent.

Problem: Is the inverse of function F in (2-8) also a function? Is F a one-to-one correspondence?

2.4 Composition

Given two functions $F: A \to B$ and $G: B \to C$, we may form a new function from A to C, called the *composite*, or *composition* of F and G, written $G \circ F$. In predicate notation function composition is defined as

(2-9) $G \circ F =_{def} \{\langle x, z \rangle \mid \text{ for some } y, \langle x, y \rangle \in F \text{ and } \langle y, z \rangle \in G\}$

Figure 2-3 shows two functions F and G and their composition.

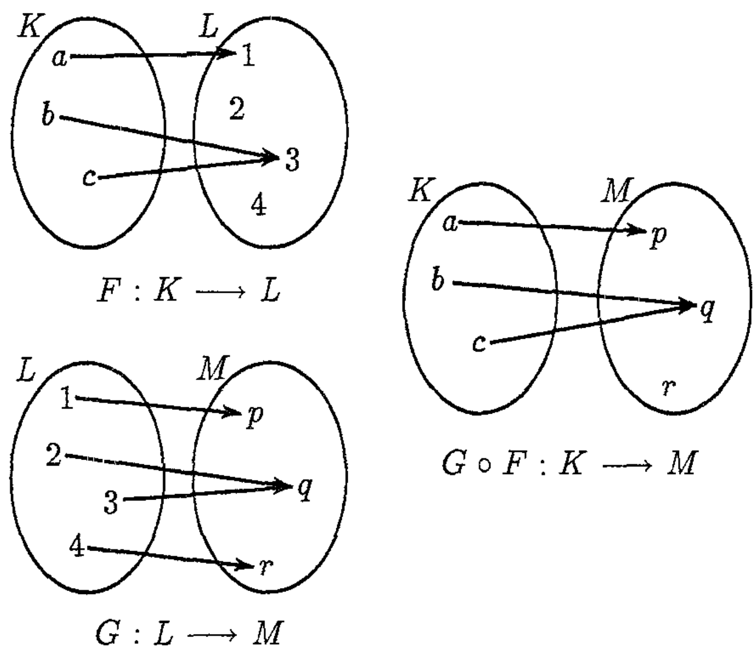

Figure 2–3: Composition of two functions F and G.

Note that F is into while G is onto and that neither is one-to-one. This shows that compositions may be formed from functions that do not have these special properties. It could happen, however, that the range of the first function is disjoint from the domain of the second, in which case, there is no y such that $\langle x, y \rangle \in F$ and $\langle y, z \rangle \in G$, and so the set of ordered pairs defined by $G \circ F$ is empty. In Figure 2-3, F is the first function and G is the second in the composition. Order is crucial here, since in general $G \circ F$ is not equal to $F \circ G$. The notation $G \circ F$ may seem to read backwards, but the value of a function F at an argument a is $F(a)$, and the value of G at the argument $F(a)$ is written $G(F(a))$. By the definition of composition, $G(F(a))$ and $(G \circ F)(a)$ produce the same value.

A function $F: A \rightarrow A$ such that $F = \{\langle x, x \rangle \mid x \in A\}$ is called the *identity function*, written id_A. This function maps each element of A to itself. Composition of a function F with the appropriate identity function gives a function that is equal to the function F itself. This is illustrated in Figure 2-4.

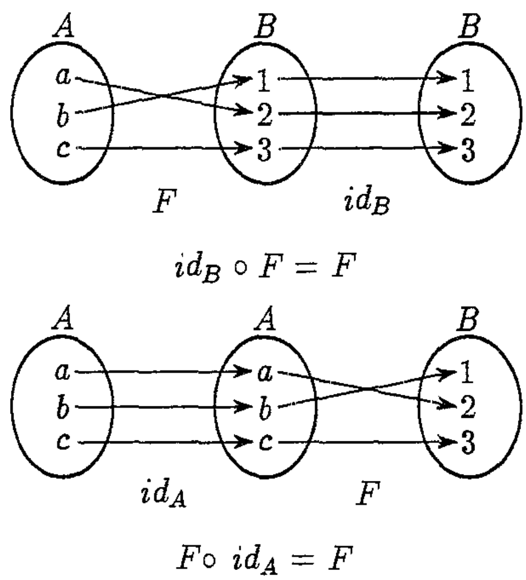

Figure 2–4: Composition with an identity function.

Given a function $F: A \rightarrow B$ that is a one-to-one correspondence (thus the inverse is also a function), we have the following general equations:

(2–10) $$\begin{aligned} F^{-1} \circ F &= id_A \\ F \circ F^{-1} &= id_B \end{aligned}$$

These are illustrated in Figure 2-5.

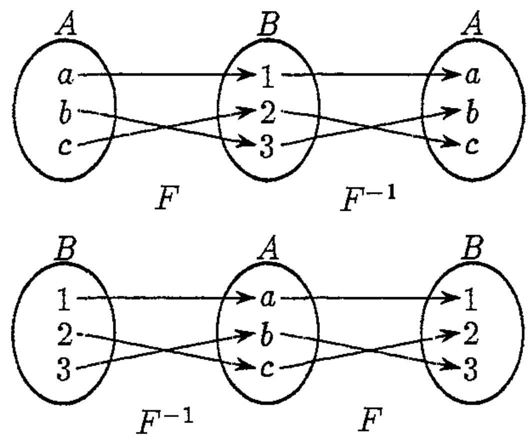

Figure 2–5: Composition of one-to-one correspondence with its inverse.

The definition of composition need not be restricted to functions but can be applied to relations in general. Given relations $R \subseteq A \times B$ and $S \subseteq B \times C$ the composite of R and S, written $S \circ R$, is the relation $\{\langle x, z \rangle \mid$ for some y, $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in S\}$ An example is shown in Figure 2-6.

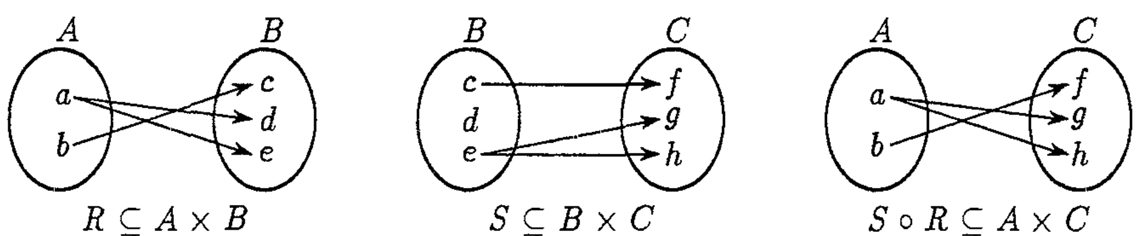

Figure 2–6: Composition of two relations R and S.

For any relation $R \subseteq A \times B$ we also have the following:

(2–11)
$$id_B \circ R = R$$
$$R \circ id_A = R$$

(Note that the identity function in A, id_A, is of course a relation and could equally well be called the identity relation in A)

The equations corresponding to (2-10) do not hold for relations (nor for functions which are not one-to-one correspondences) However, we have for any *one-to-one* relation $R: A \to B$:

$$(2\text{–}12) \quad \begin{array}{lcl} R^{-1} \circ R & \subseteq & id_A \\ R \circ R^{-1} & \subseteq & id_B \end{array}$$

We should note here that our previous remarks about ternary, quaternary, etc. relations can also be carried over to functions. A function may have as its domain a set of ordered n-tuples for any n, but each such n-tuple will be mapped into a unique value in the range. For example, there is a function mapping each pair of natural numbers into their sum.

Exercises

1. Let $A = \{b, c\}$ and $B = \{2, 3\}$

   (a) Specify the following sets by listing their members.

(i)	$A \times B$	(iv)	$(A \cup B) \times B$
(ii)	$B \times A$	(v)	$(A \cap B) \times B$
(iii)	$A \times A$	(vi)	$(A - B) \times (B - A)$

   (b) Classify each statement as true or false.

   (i) $(A \times B) \cup (B \times A) = \emptyset$

   (ii) $(A \times A) \subseteq (A \times B)$

   (iii) $\langle c, c \rangle \subseteq (A \times A)$

   (iv) $\{\langle b, 3 \rangle, \langle 3, b \rangle\} \subseteq (A \times B) \cup (B \times A)$

   (v) $\emptyset \subseteq A \times A$

   (vi) $\{\langle b, 2 \rangle, \langle c, 3 \rangle\}$ is a relation from A to B

   (vii) $\{\langle b, b \rangle\}$ is a relation in A

   (c) Consider the following relation from A to $(A \cup B)$:
   $R = \{\langle b, b \rangle, \langle b, 2 \rangle, \langle c, 2 \rangle, \langle c, 3 \rangle\}$

   (i) Specify the domain and range of R

   (ii) Specify the complementary relation R' and the inverse R^{-1}

   (iii) Is $(R')^{-1}$ (the inverse of the complement) equal to $(R^{-1})'$ (the complement of the inverse)?

2. Let $A = \{a, b, c\}$ and $B = \{1, 2\}$. How many distinct relations are there from A to B? How many of these are functions from A to B? How many of the functions are onto? one-to-one? Do any of the functions have inverses that are functions? Answer the same questions for all relations from B to A.

3. Let

$$R_1 = \{\langle 1,1 \rangle, \langle 2,1 \rangle, \langle 3,4 \rangle, \langle 2,2 \rangle, \langle 3,3 \rangle, \langle 4,4 \rangle, \langle 4,1 \rangle\}$$

$$R_2 = \{\langle 3,4 \rangle, \langle 1,2 \rangle, \langle 1,4 \rangle, \langle 2,3 \rangle, \langle 2,4 \rangle, \langle 1,3 \rangle\}$$

(both relations in A, where $A = \{1, 2, 3, 4\}$).

(a) Form the composites $R_2 \circ R_1$ and $R_1 \circ R_2$. Are they equal?

(b) Show that $R_1^{-1} \circ R_1 \neq id_A$ and that $R_2^{-1} \circ R_2 \not\subseteq id_A$.

4. For the functions F and G in Figure 2-3:

(a) show that $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$.

(b) Show that the corresponding equation holds for relations R and S in Figure 2-6.

Chapter 3

Properties of Relations

3.1 Reflexivity, symmetry, transitivity, and connectedness

Certain properties of binary relations are so frequently encountered that it is useful to have names for them. The properties we shall consider are *reflexivity*, *symmetry*, *transitivity*, and *connectedness*. All these apply only to relations *in* a set, i.e., in $A \times A$ for example, not to relations from A to B, where $B \neq A$.

Reflexivity

Given a set A and a relation R in A, R is *reflexive* if and only if all the ordered pairs of the form $\langle x, x \rangle$ are in R for every x in A.

As an example, take the set $A = \{1, 2, 3\}$ and the relation $R_1 = \{\langle 1,1 \rangle, \langle 2,2 \rangle, \langle 3,3 \rangle, \langle 3,1 \rangle\}$ in A. R_1 is reflexive because it contains the ordered pairs $\langle 1,1 \rangle, \langle 2,2 \rangle$, and $\langle 3,3 \rangle$. The relation $R_2 = \{\langle 1,1 \rangle, \langle 2,2 \rangle\}$ is nonreflexive since it lacks the ordered pair $\langle 3,3 \rangle$ and thus fails to meet the definitional requirement that it contains the ordered pair $\langle x, x \rangle$ *for every x in A*. Another way to state the definition of reflexivity is to say that a relation R in A is reflexive if and only if id_A, the identity relation in A, is a subset of R. The relation 'has the same birthday as' in the set of human beings is reflexive.

A relation which fails to be reflexive is called nonreflexive, but if it contains *no* ordered pair $\langle x, x \rangle$ with identical first and second members, it is said to be *irreflexive*. $R_3 = \{\langle 1,2 \rangle, \langle 3,2 \rangle\}$ is an example of an irreflexive relation in A. Irreflexivity is a stronger condition than nonreflexivity since every irreflexive relation is nonreflexive but not conversely. The relation 'is taller than' in the set of human beings is irreflexive (therefore also nonreflexive), while the relation 'is a financial supporter of' is nonreflexive (but not irreflexive, since some people are financially self-supporting) Note that a relation R in A is nonreflexive if and only if $id_A \not\subseteq R$; it is irreflexive if and only if $R \cap id_A = \emptyset$.

Symmetry

Given a set A and a binary relation R in A, R is *symmetric* if and only if for every ordered pair $\langle x, y \rangle$ in R, the pair $\langle y, x \rangle$ is also in R. It is important to note that this definition does not require every ordered pair of $A \times A$ to be in R. Rather for a relation R to be symmetric it must always be the case that *if* an ordered pair is in R, *then* the pair with the members reversed is also in R.

Here are some examples of symmetric relations in $\{1, 2, 3\}$:

(3–1) $\{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 3,2 \rangle, \langle 2,3 \rangle\}$
$\{\langle 1,3 \rangle, \langle 3,1 \rangle\}$
$\{\langle 2,2 \rangle\}$

$\{\langle 2,2 \rangle\}$ is a symmetric relation because for every ordered pair in it, i.e., $\langle 2,2 \rangle$, it is true that the ordered pair with the first and second members reversed, i.e., $\langle 2,2 \rangle$, is in the relation. Another example of a symmetric relation is 'is a cousin of' on the set of human beings. If for some $\langle x, y \rangle$ in R, the pair $\langle y, x \rangle$ is not in R then R is *nonsymmetric*. The relation 'is a sister of' on the set of human beings is nonsymmetric (since the second member may be male. It is, however, a symmetric relation defined on the set of human females).

The following relations in $\{1, 2, 3\}$ are nonsymmetric:

(3–2) $\{\langle 2,3 \rangle, \langle 1,2 \rangle\}$
$\{\langle 3,3 \rangle, \langle 1,3 \rangle\}$
$\{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle, \langle 1,1 \rangle, \langle 2,3 \rangle\}$

If it is *never* the case that for any $\langle x, y \rangle$ in R, the pair $\langle y, x \rangle$ is in R, then the relation is called *asymmetric*. The relation 'is older than' is asymmetric on the set of human beings. Note that an asymmetric relation must be irreflexive (because nothing in the asymmetry definition requires x and y to be distinct). The following are examples of asymmetric relations in $\{1, 2, 3\}$:

(3–3) $\{\langle 2,3 \rangle, \langle 1,2 \rangle\}$
$\{\langle 1,3 \rangle, \langle 2,3 \rangle, \langle 1,2 \rangle\}$
$\{\langle 3,2 \rangle\}$

A relation is *anti-symmetric* if whenever both $\langle x,y \rangle$ and $\langle y,x \rangle$ are in R, then $x = y$. This definition says only that *if* both $\langle x,y \rangle$ and $\langle y,x \rangle$ are in R, then x and y are identical; it does not require $\langle x,x \rangle \in R$ for all $x \in A$. In other words, the relation need not be reflexive in order to be anti-symmetric.

The following relations in $\{1,2,3\}$ are anti-symmetric.

(3–4) $\{\langle 2,3 \rangle, \langle 1,1 \rangle\}$
$\{\langle 1,1 \rangle, \langle 2,2 \rangle\}$
$\{\langle 1,2 \rangle, \langle 2,3 \rangle\}$

Transitivity

A relation R is *transitive* if and only if for all ordered pairs $\langle x,y \rangle$ and $\langle y,z \rangle$ in R, the pair $\langle x,z \rangle$ is also in R.

Because there is no necessity for x, y, and z all to be distinct, the following relation meets the definition of transitivity,

(3–5) $\{\langle 2,2 \rangle\}$

where $x = y = z = 2$.

The relation given in (3-6) is not transitive,

(3–6) $\{\langle 2,3 \rangle, \langle 3,2 \rangle, \langle 2,2 \rangle\}$

because $\langle 3,2 \rangle$ and $\langle 2,3 \rangle$ are members, but $\langle 3,3 \rangle$ is not.

Here are some more examples of transitive relations:

(3–7) $\{\langle 1,2 \rangle, \langle 2,3 \rangle, \langle 1,3 \rangle\}$
$\{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 1,1 \rangle, \langle 2,2 \rangle\}$
$\{\langle 1,2 \rangle, \langle 2,3 \rangle, \langle 1,3 \rangle, \langle 3,2 \rangle, \langle 2,1 \rangle, \langle 3,1 \rangle, \langle 1,1 \rangle, \langle 2,2 \rangle, \langle 3,3 \rangle\}$

The relation 'is an ancestor of' is transitive in the set of human beings. If a relation fails to meet the definition of transitivity, it is *nontransitive*. If for no pairs $\langle x,y \rangle$ and $\langle y,z \rangle$ in R, the ordered pair $\langle x,z \rangle$ is in R, then the relation is *intransitive*. For example, the relation 'is the mother of' in the set of human beings is intransitive.

Relation (3-6) is nontransitive, as are the following two:

(3–8) $\{\langle 1,2\rangle,\langle 2,3\rangle\}$
$\{\langle 1,2\rangle,\langle 2,3\rangle,\langle 1,3\rangle,\langle 3,1\rangle\}$

The first of these relations is also intransitive, as are the following relations:

(3–9) $\{\langle 3,1\rangle,\langle 1,2\rangle,\langle 2,3\rangle\}$
$\{\langle 3,2\rangle,\langle 1,3\rangle\}$

Connectedness

A relation R in A is *connected* (or *connex*) if and only if for every two *distinct* elements x and y in A, $\langle x,y\rangle \in R$ or $\langle y,x\rangle \in R$ (or both).

Note that the definition of connectedness refers, as does the definition of reflexivity, to all the members of the set A. Further, the pairs $\langle x,y\rangle$ and $\langle y,x\rangle$ mentioned in the definition are explicitly specified as containing nonidentical first and second members. Pairs of the form $\langle x,x\rangle$ are not prohibited in a connected relation, but they are irrelevant in determining connectedness.

The following relations in $\{1,2,3\}$ are connected:

(3–10) $\{\langle 1,2\rangle,\langle 3,1\rangle,\langle 3,2\rangle\}$
$\{\langle 1,1\rangle,\langle 2,3\rangle,\langle 1,2\rangle,\langle 3,1\rangle,\langle 2,2\rangle\}$

The following relations in $\{1,2,3\}$, which fail the definition, are nonconnected.

(3–11) $\{\langle 1,2\rangle,\langle 2,3\rangle\}$
$\{\langle 1,3\rangle,\langle 3,1\rangle,\langle 2,2\rangle,\langle 3,2\rangle\}$

It may be useful at this point to give some examples of relations specified by predicates and to consider their properties of reflexivity, symmetry, transitivity, and connectedness

(3–12) *Example*: R_f is the relation 'is father of' in the set H of all human beings. R_f is irreflexive (no one is his own father); asymmetric (if x is y's father, then it is never true that y is x's father); intransitive (if x is y's father and y is z's father, then x is z's grandfather but not z's father); and nonconnected (there are distinct individuals x and y in H such that neither 'x is the father of y' nor 'y is the father of x' is true).

(3–13) *Example*: R is the relation 'greater than' defined in the set $Z = \{1, 2, 3, 4, \ldots\}$ of all the positive integers. Z contains an infinite number of members and so does R, but we are able to determine the relevant properties of R from our knowledge of the properties of numbers in general. R is irreflexive (no number is greater than itself); asymmetric (if $x > y$, then $y \not> x$; transitive (if $x > y$ and $y > z$, then $x > z$), and connected (for every distinct pair of integers x and y, either $x > y$ or $y > x$.

(3–14) *Example*: R_a is the relation defined by 'x is the same age as y,' in the set H of all living human beings. R_a is reflexive (everyone is the same age as himself or herself); symmetric (if x is the same age as y, then y is the same age as x); transitive (if x and y are the same age and so are y and z, then x is the same age as z); and nonconnected (there are distinct individuals in H who are not of the same age).

3.2 Diagrams of relations

It may be helpful in assimilating the notions of reflexivity, symmetry and transitivity to represent them in relational diagrams. The members of the relevant set are represented by labeled points (the particular spatial arrangement of them is irrelevant). If x is related to y, i.e. $\langle x, y \rangle \in R$, an arrow connects the corresponding points. For example,

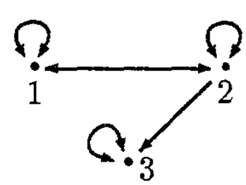

Figure 3–1: Relational diagram.

Figure 3-1 represents the relation

$$R = \{\langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle, \langle 1, 1 \rangle, \langle 2, 3 \rangle, \langle 3, 3 \rangle\}$$

It is apparent from the diagram that the relation is reflexive, since every point bears a loop. The relation is nonsymmetric since 3 is not related to 2 whereas 2 is related to 3. It cannot be called asymmetric or antisymmetric, however, since 1 is related to 2 and 2 is related to 1. It is nontransitive since 1 is related to 2 and 2 is related to 3, but there is no direct arrow from 1 to 3. The relation cannot be intransitive because of the presence of pairs such as $\langle 1, 1 \rangle$.

If a relation is connected, every pair of distinct points in its diagram will be directly joined by an arrow. We see that R is no connected since there is not direct connection between 1 and 3 in Figure 3-1.

3.3 Properties of inverses and complements

Given that a relation R has certain properties of reflexivity, symmetry, transitivity or connectedness, one can often make general statements about the question whether these properties are preserved when the inverse R^{-1} or complement R' of that relation is formed.

For example, take a reflexive relation R in A. By the definition of reflexive relations, for every $x \in A$, $\langle x, x \rangle \in R$. Since R^{-1} has all the ordered pairs of R, but with the first and second members reversed, then every pair $\langle x, x \rangle$ is also in R^{-1}. So the inverse of R is reflexive also. The complement R' contains all ordered pairs in $A \times A$ that are not in R. Since R contains every pair of the form $\langle x, x \rangle$ for any $x \in A$, R' contains none of them. The complement relation is therefore irreflexive.

As another example, take a symmetric relation R in A. Does its complement have this property? Let's assume that the complement R' is not symmetric, and see what we can derive from that assumption. If R' is not symmetric, then there is some $\langle x, y \rangle \in R'$ such that $\langle y, x \rangle \notin R'$, by the definition of a nonsymmetric relation. Since $\langle y, x \rangle \notin R'$, $\langle y, x \rangle$ must be in the complement of R', which is R itself. Because R is symmetric, $\langle x, y \rangle$ must also be in R. But one and the same ordered pair $\langle x, y \rangle$ cannot be both in R *and* in its complement R', so the assumption that the complement R' is not symmetric leads to an absurd conclusion. That means that the assumption cannot be true and the complement R' must be symmetric after all. If R is a symmetric relation in A, then the complement R' is symmetric and vice versa (the latter follows from essentially the same reasoning with R' substituted for R). This mode of reasoning is an instance of what is called a *reductio ad absurdum* proof in logic. It is characterized by making an assumption which leads to a necessarily false conclusion; you may then conclude that the negation of that assumption is true. In Chapter 6 we will introduce rules of inference which will allow such arguments to be made completely precise.

For sake of easy reference the table in Figure 3-2 presents a summary of properties of relations and those of their inverses and complements. These can all be proved on the basis of the definitions of the concepts and the laws of set theory. Since we have not yet introduced a formal notion of proof, we will not offer proofs here, but it is a good exercise to convince yourself of the facts by trying out a few examples, reasoning informally along the lines illustrated above.

R (not $\emptyset$)	R^{-1}	R'
reflexive	reflexive	irreflexive
irreflexive	irreflexive	reflexive
symmetric	symmetric ($R^{-1} = R$)	symmetric
asymmetric	asymmetric	non-symmetric
antisymmetric	antisymmetric	depends on R
transitive	transitive	depends on R
intransitive	intransitive	depends on R
connected	connected	depends on R

Figure 3–2: Preservation of properties of a relation in its inverse and its complement.

3.4 Equivalence relations and partitions

An especially important class of relations are the *equivalence relations.* They are relations which are reflexive, symmetric and transitive. Equality is the most familiar example of an equivalence relation. Other examples are 'has the same hair color as', and 'is the same age as'. The use of equivalence relations on a domain serves primarily to structure a domain into subsets whose members are regarded as equivalent with respect to that relation.

For every equivalence relation there is a natural way to divide the set on which it is defined into mutually exclusive (disjoint) subsets which are called *equivalence classes*. We write $[\![x]\!]$ for the set of all y such that $\langle x, y \rangle \in R$.

Thus, when R is an equivalence relation, $[\![x]\!]$ is the equivalence class which contains x The relation 'is the same age as' divides the set of people into age groups, i.e., sets of people of the same age Every pair of distinct equivalence classes is disjoint, because each person, having only one age, belongs to exactly one equivalence class. This is so even when somebody is 120 years old, and is the only person of that age, consequently occupying an equivalence class all by himself. By dividing a set into mutually exclusive and collectively exhaustive nonempty subsets we effect what is called a *partitioning* of that set.

Given a non-empty set A, a *partition* of A is a collection of non-empty subsets of A such that (1) for any two distinct subsets X and Y, $X \cap Y = \emptyset$ and (2) the union of all the subsets in the collection equals A The notion of a partition is not defined for an empty set. The subsets that are members of a partition are called *cells* of that partition.

For example, let $A = \{a, b, c, d, e\}$. Then, $P = \{\{a, c\}, \{b, e\}, \{d\}\}$ is a partition of A because every pair of cells is disjoint: $\{a, c\} \cap \{b, e\} = \emptyset$, $\{b, e\} \cap \{d\} = \emptyset$, and $\{a, c\} \cap \{d\} = \emptyset$; and the union of all the cells equals A: $\bigcup\{\{a, c\}, \{b, e\}, \{d\}\} = A$.

The following three sets are also partitions of A:

(3–15)
$$P_1 = \{\{a, c, d\}, \{b, e\}\}$$
$$P_2 = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}\}$$
$$P_3 = \{\{a, b, c, d, e\}\}$$

P_3 is the trivial partition of A into only one set. Note however that the definition of a partition is satisfied.

The following two sets are not partitions of A:

(3–16)
$$C = \{\{a, b, c\}, \{b, d\}, \{e\}\}$$
$$D = \{\{a\}, \{b, e\}, \{c\}\}$$

C fails the definition because $\{a, b, c\} \cap \{b, d\} \neq \emptyset$ and D because $\bigcup\{\{a\}, \{b, e\}, \{c\}\} \neq A$

There is a close correspondence between partitions and equivalence relations. Given a partition of set A, the relation $R = \{\langle x, y \rangle \mid x$ and y are in the same cell of the partition$\}$ is an equivalence relation. Conversely, given a reflexive, symmetric, and transitive relation R in A, there exists a partition of A in which x and y are in the same cell if and only if x and y are related by

R. The equivalence classes specified by R are just the cells of the partition. An equivalence relation in A is sometimes said to *induce a partition of A*.

As an example, consider the set $A = \{1,2,3,4,5\}$ and the equivalence relation

(3–17) $R = \{\langle 1,1\rangle, \langle 1,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle, \langle 2,2\rangle, \langle 2,4\rangle, \langle 4,2\rangle, \langle 4,5\rangle,$
$\langle 4,4\rangle, \langle 5,2\rangle, \langle 5,4\rangle, \langle 5,5\rangle, \langle 2,5\rangle\}$

which the reader can verify to be reflexive, symmetric, and transitive. In this relation 1 and 3 are related among themselves in all possible ways, as are 2, 4, and 5, but no members of the first group are related to any member of the second group. Therefore, R defines the equivalence classes $\{1,3\}$ and $\{2,4,5\}$, and the corresponding partition induced on A is

(3–18) $P_R = \{\{1,3\},\{2,4,5\}\}$

Given a partition such as

(3–19) $Q = \{\{1,2\},\{3,5\},\{4\}\}$

the relation R_Q consisting of all ordered pairs $\langle x, y\rangle$ such that x and y are in the same cell of the partition is as follows:

(3–20) $R_Q = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 3,5\rangle, \langle 5,3\rangle, \langle 5,5\rangle, \langle 4,4$

R_Q is seen to be reflexive, symmetric, and transitive, and it is thus an equivalence relation.

Another example is the equivalence relation 'is on the same continent as' on the set A = {France, Chile, Nigeria, Ecuador, Luxembourg, Zambia, Ghana, San Marino, Uruguay, Kenya, Hungary}. It partitions A into three equivalence classes: (1) A_1 = {France, Luxembourg, San Marino, Hungary}, (2) A_2 = {Chile, Ecuador, Uruguay} and (3) A_3 = {Nigeria, Zambia, Ghana, Kenya}.