LOGIC

The Laws of Truth

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16

Set Theory

This chapter—more in the nature of an appendix—explains basic concepts from set theory, some of which have been employed earlier in this book; it is not a full introduction to the field of set theory.

16.1 Sets

A *set* is a collection of objects. These objects are said to be *members* or *elements* of the set, and the set is said to *contain* these objects.

If we are in a position to name all elements of a set, we can name the set itself by putting braces ("{" and "}") around them. For example, we denote the set containing the numbers 1, 2, and 3 as {1, 2, 3} and the set containing Alice, Bob, and Carol as {Alice, Bob, Carol}. If we cannot name all elements of a set, we might do one of two things. If the elements come in some known order, we can name the first few of them and then write an ellipsis (". . ."). For example, we denote the set of all positive integers as {1, 2, 3, . . .} and the set of all even positive integers as {2, 4, 6, . . .}. Alternatively, we can state a condition C that is satisfied by all and only the elements of the set, and we then denote the set as $\{x : C\}$ (or $\{x|C\}$). For example, the set of all red things is denoted $\{x : x \text{ is red}\}$ (read as "the set of all x such that x is red"), and the set of all even numbers is denoted $\{x : x \text{ is even}\}$ (read as "the set of all x such that x is even").

We use the symbol $\in$ (epsilon) to denote membership, as in $1 \in \{1, 2, 3\}$ and Alice $\in$ {Alice, Bob, Carol}. To say that something is not a member of a set, we use the symbol $\notin$, as in $4 \notin \{1, 2, 3\}$ and Dave $\notin$ {Alice, Bob, Carol}. The symbol $\in$ is a two-place relation symbol, but as with $=$, we write it in between its arguments (as in $x \in S$), not in front of them. The expression $x \notin S$ can be seen as an abbreviation for $\neg x \in S$.

When asked to picture the set containing, say, Alice and Bob, many people will simply picture Alice and Bob standing side by side. This isn't the best way to think of sets. Alice and Bob are the members of the set containing Alice

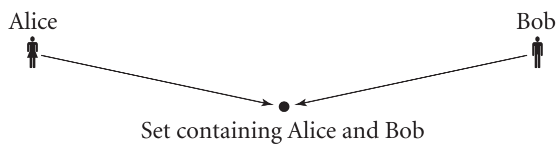


Figure 16.1. Alice, Bob, and the set that contains them.

and Bob, but the set itself is a third thing, distinct from its two members. So we should picture the situation as in Figure 16.1, where the arrows indicate membership (i.e., the thing at the tail of an arrow is a member of the thing at the head of that arrow). This is the guiding idea behind set theory: to treat a collection of objects—that is, a set—as an object in its own right. Set theory is then the theory of these objects—of sets. As Georg Cantor—the founder of set theory—put it: a set is a *many* or *multiplicity* that can be conceived of as *one* or *single.*[1] Note that—unlike its members, Alice and Bob—the set containing Alice and Bob is not visible or tangible. For this reason sets are often referred to as *abstract* objects.

There is a set called the *empty set* or *null set,* symbolized by $\emptyset$, which has no elements. This may sound odd. A set is supposed to be a collection of things—but we cannot collect together nothing! So how can there be a set containing no things? Actually, the idea makes perfect sense, once we think of it in the right way—that is, once we remember to think of sets as objects, distinct from their members, with membership indicated by arrows (as in Figure 16.1). We then picture the empty set (i.e., the set with no members) as a dot—an object, a thing, just like all other sets—that simply has no arrows pointing to it.

16.1.1 Extensionality

Suppose we have some kind of thing: Ps. We make a first choice of a P—call it x. We make a second choice of a P (maybe a different thing from our first choice, or maybe we have chosen the same thing a second time)—call it y. An *identity condition* for Ps determines whether $x = y$; that is, whether we chose the same thing twice or chose two different things, for any choices x and y. Sets have a very simple identity condition: for any sets x and y, x and y are identical (i.e., $x = y$) iff every member of x is a member of y, and vice versa. This property of sets—that they are individuated by their members; that if "two" sets have exactly the same members, then they are in fact one and the same set—is known as *extensionality.*

Here are some examples:

$$\{1, 2\} = \{2, 1\}$$

The set on the left has two elements (1 and 2), and each of them is a member of the set on the right. The set on the right has two elements (2 and 1), and each of them is a member of the set on the left. Thus, every member of the set on the left is a member of the set on the right, and vice versa, so they are two different ways of writing the same set. When we name a set by listing its members with braces around them, the order in which we write the elements of the set within the braces does not matter.

$$\{1\} = \{1, 1\}$$

The set on the left has one element (1), and it is a member of the set on the right. The set on the right has just one element (1)—we have simply named this element twice when writing the set on the right—and it is a member of the set on the left. Thus, every member of the set on the left is a member of the set on the right, and vice versa, so they are two different ways of writing the same set. When we name a set by listing its members with braces around them, it makes no difference whether we write a given element once or multiple times: the only significant thing is whether a certain object is named as an element at all.

$$\{4\} = \{2 + 2\}$$

The expressions "$2 + 2$" and "4" pick out the same number: thus, the only element of the set on the left is a member of the set on the right, and vice versa. Note here that even though extensionality fixes the facts as to whether set x is identical to set y, for any sets x and y, it need not always enable us to see whether sets x and y are identical. For example, because $2 + 2$ and 4 are the same number, extensionality fixes that the set $\{4\}$ is the same object as the set $\{2 + 2\}$. However, if someone does not know that $2 + 2 = 4$, then simply knowing the principle of extensionality will not enable him to see that $\{4\}$ and $\{2 + 2\}$ (described thus) are the same set.

$$\{2, 4, 6, \ldots\} = \{x : x \text{ is an even positive integer}\}$$

Again, the expressions on the left and right of the identity sign are just two different ways of writing the same set.

Properties—in contrast with sets—are *intensional.* Consider a property, say, the property of redness. The set of all things that possess a property is often called the *extension* of the property; thus, the set containing all and only red things is the extension of the property of redness.[2] Now two distinct properties might be possessed by exactly the same objects; that is, they might have the same extension. For example, the property of being a human being is not (intuitively) the same as the property of being a featherless biped, but both properties have the same extension (i.e., all humans are featherless bipeds and vice versa). Thus, we say that properties are intensional, as opposed to extensional: knowing that properties P and Q are possessed by the same objects does not allow you to conclude that properties P and Q are identical, whereas knowing that sets S and T contain the same objects does allow you to conclude that S and T are identical. So "being possessed by the same objects" is not the identity condition for properties (whereas "containing the same objects" is the identity condition for sets). In fact there is no obviously correct precise identity condition for properties. Certain cases might be clear enough—such as the featherless biped example—but there is no widely accepted theory spelling out a general precise identity condition for properties. One of the advantages of working with sets—rather than properties—is their crystal clear identity condition (i.e., extensionality).

16.1.2 Subsets

A set S is a subset of a set T—in symbols, $S \subseteq T$—iff every member of S is a member of T:

$$S \subseteq T \text{ iff } \forall x(x \in S \rightarrow x \in T) \tag{16.1}$$

Note that this definition leaves open whether or not $S = T$: that depends upon whether there is anything in T that is not in S. If there is nothing in T that is not in S (i.e., if $T \subseteq S$ as well as $S \subseteq T$), then $S = T$. This is just the principle of extensionality phrased in a new way. If there is something in T that is not in S (i.e., $S \subseteq T$ but not $T \subseteq S$), then S is a *proper* subset of T, symbolized by $S \subsetneq T$.[3] Note that every set is (trivially) a subset of itself, but no set is a proper subset of itself.

The null set is a subset of every set. Given any set T, it is automatically true—because $\emptyset$ has no members—that every member of $\emptyset$ is a member of T. Recall (Exercises 9.4.3, question 5(i)) that "all Fs are Gs" is true when there are no Fs. Similarly, because $x \in \emptyset$ is false for every x, the following comes out true no matter what set T is:

$$\forall x(x \in \emptyset \rightarrow x \in T)$$

But this is just the condition required for $\emptyset$ to be a subset of T; hence, for all T, $\emptyset \subseteq T$.

We can now see that the empty set is unique; that is, there is only one empty set (there are not two different sets, each of which has no members). For suppose there were two empty sets, a and b. For the reasons just given, $a \subseteq b$ and $b \subseteq a$—but then, by extensionality, $a = b$.

A set containing just one element—for example, {3}—is a *singleton* or *unit set.*

Note that 1 is an *element* of the set $\{1, 2, 3\}$ but is not a subset of it, whereas $\{1\}$ is a subset of the set $\{1, 2, 3\}$ but is not an element of it. Sometimes we are given a set S and we want to consider a set of subsets of S. For example, suppose we have the set $S = \{1, 2, 3, 4\}$, and we want to consider the set S_2 of all two-membered subsets of this set:

$$S_2 = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$$

(Because $\{2, 3\} = \{3, 2\}$, we do not list $\{3, 2\}$ separately. Similarly for $\{2, 1\}$ etc.) Note that:

$$\{1, 2\} \subseteq S$$
$$\{1, 2\} \in S_2$$

That is, an element of S_2 is a subset of S.

One very important set of subsets of any set S is the *power set* of S—the set of *all* subsets of S—symbolized by $\wp S$:

$$\wp S = \{x : x \subseteq S\}$$

For example, for $S = \{1, 2, 3\}$,

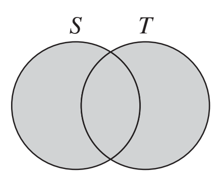
$$\wp S = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

16.1.3 Operations on Sets

The *union* of two sets S and T, denoted $S \cup T$, contains everything in either S or T (or both):

$$S \cup T = \{x : x \in S \vee x \in T\}$$

or visually:

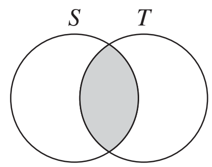


Here, the left circle represents the set S (i.e., think of the members of S as the things within this circle; note that these elements of S are not shown in the picture); the right circle represents the set T; the union of S and T is shaded gray.

The *intersection* of two sets S and T, denoted $S \cap T$, contains everything which is in both S and T:

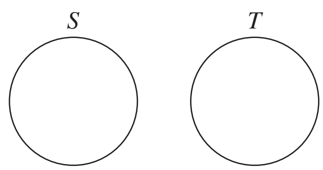
$$S \cap T = \{x : x \in S \wedge x \in T\}$$

or visually:



Here, the intersection of S and T is shaded gray.

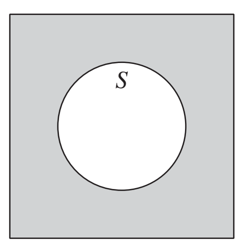
Two sets S and T are *disjoint* if they have no members in common; that is, if $S \cap T = \emptyset$:



Often when dealing with some sets, it is useful to consider them as subsets of some background set (e.g., the background set might be the domain of some model). The *complement* of a set S, denoted S', is the set of all things that are *not* in S.[4] Here it is important that we are restricting ourselves to the contents of some background set: S' contains everything in the background set that is not in S, not everything at all that is not in S:

$$S' = \{x : \neg x \in S\}$$

or visually:

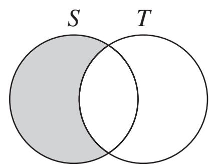


Here, the square represents the background set; the circle represents the set S; the complement of S is shaded gray.

The set-theoretic *difference* of two sets S and T (taken in that order), denoted $S \setminus T$, is the set of things in S but not in T:

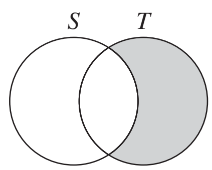
$$S \setminus T = \{x : x \in S \wedge \neg x \in T\}$$

or visually:



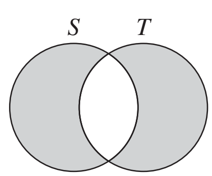
Here, the shaded area is $S \setminus T$. The set $S \setminus T$ is also known as the *relative complement* of T in S. Note that if we think of S and T as subsets of a background set U, then $S \setminus T = S \cap T'$, and $S' = U \setminus S$.[5]

Note that, for any sets S and T, $S \cup T = T \cup S$ and $S \cap T = T \cap S$.[6] It is not the case, however, that for any sets S and T, $S \setminus T = T \setminus S$. Compare the following picture of $T \setminus S$ to the previous picture of $S \setminus T$:



Of course, if $S = T$ then $S \setminus T = T \setminus S = \emptyset$.[7]

There is an evident parallel between the set-theoretic operations of complement, union, and intersection and the logical operations of negation, disjunction, and conjunction, respectively: the complement of S contains all objects *not* in S; the union of S and T contains all objects in S *or* in T; the intersection of S and T contains all objects in S *and* in T. Recall (§6.6) that every possible two-place connective can be defined in terms of $\neg$, $\vee$, and $\wedge$. Similarly, suppose we have two sets, S and T, that are subsets of a background set U. Suppose we want to specify a third subset, V, such that for any object x in U, whether x is in V is completely determined by whether x is in S and whether x is in T. Then, any such V can be defined in terms of S and T and the operations of complement, union, and intersection. We have already seen an example of this: $S \setminus T = S \cap T'$. Here is a second example. The *symmetric difference* of two sets S and T, denoted $S \Delta T$, contains everything that is in exactly one of S and T:



This set may be defined in any of the following ways:

$$S \Delta T = (S \cup T) \setminus (S \cap T)$$
$$S \Delta T = (S \setminus T) \cup (T \setminus S)$$
$$S \Delta T = (S \cup T) \cap (S \cap T)'$$

The third is a direct definition in terms of S and T and the operations of complement, union, and intersection. The first two reduce to such definitions when we define out the relative complement operation in these terms. Note that, unlike the set-theoretic difference operation, the symmetric difference operation is symmetric (hence its name); that is, for any sets S and T, $S \Delta T = T \Delta S$.

You may notice that the symmetric difference operation is the set-theoretic analogue of exclusive disjunction (§6.4). That is, we could specify the symmetric difference of S and T as:

$$S \Delta T = \{x : x \in S \veebar x \in T\}$$

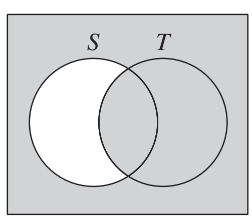
If we take any other two-place connective, we can likewise obtain a corresponding operation on sets. For example, corresponding to the conditional, we could specify an operation $\stackrel{s}{\rightarrow}$ on sets as follows (we put an "s" on top of the arrow symbol to indicate that this new operation takes sets as arguments, whereas the conditional $\rightarrow$ connects wffs):

$$S \stackrel{s}{\rightarrow} T = \{x : x \in S \rightarrow x \in T\}$$

Remembering that $\alpha \rightarrow \beta$ is equivalent to $\neg\alpha \vee \beta$ and to $\neg(\alpha \wedge \neg\beta)$, the set $S \stackrel{s}{\rightarrow} T$ could be defined in either of the following ways:

$$S \stackrel{s}{\rightarrow} T = S' \cup T$$
$$S \stackrel{s}{\rightarrow} T = (S \cap T')'$$

As you can see by comparing the following picture of $S \stackrel{s}{\rightarrow} T$ with the earlier picture of $S \setminus T$, $S \stackrel{s}{\rightarrow} T = (S \setminus T)'$.



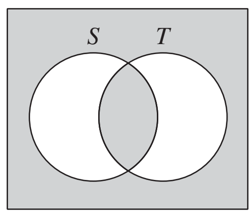
To take a second example, corresponding to the biconditional, we could specify an operation $\stackrel{s}{\leftrightarrow}$ on sets as:

$$S \stackrel{s}{\leftrightarrow} T = \{x : x \in S \leftrightarrow x \in T\}$$

Remembering that $\alpha \leftrightarrow \beta$ is equivalent to $(\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta)$ and to $\neg((\alpha \vee \beta) \wedge \neg(\alpha \wedge \beta))$, the set $S \stackrel{s}{\leftrightarrow} T$ could be defined in either of the following ways:

$$S \stackrel{s}{\leftrightarrow} T = (S \cap T) \cup (S' \cap T')$$
$$S \stackrel{s}{\leftrightarrow} T = ((S \cup T) \cap (S \cap T)')'$$

As you can see by comparing the following picture of $S \stackrel{s}{\leftrightarrow} T$ with the earlier one of $S\Delta T$, $S \stackrel{s}{\leftrightarrow} T = (S\Delta T)'$. Also, $S \stackrel{s}{\leftrightarrow} T = (S \stackrel{s}{\rightarrow} T) \cap (T \stackrel{s}{\rightarrow} S)$. You can see this identity by looking at the picture of $S \stackrel{s}{\rightarrow} T$, imagining a picture of $T \stackrel{s}{\rightarrow} S$, and comparing them with the following picture of $S \stackrel{s}{\leftrightarrow} T$, or by noting that $\alpha \leftrightarrow \beta$ is equivalent to $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$.



16.1.4 What Sets Exist?

An intuitively appealing principle is that every property has an extension: for any property, there is a set of objects that have that property. (It may be the empty set, but that is still a set.) We can make this idea more precise by replacing the notion of "property" with that of a condition specifiable in a particular formal language. Let's take the fragment of GPLI including no nonlogical symbols (no names, and no predicates apart from =) and add the set-theoretic symbol $\in$ (a two-place predicate). Call the resulting language GPLI with Set Membership (GPLIS). An open formula $\alpha(x)$ of GPLIS—which contains free occurrences of the variable x—can be thought of as a condition that objects may or may not satisfy. Now the more precise version of the intuitive thought is that for any such condition $\alpha(x)$, there exists a set containing all and only the objects satisfying the condition—that is, the set:

$$\{x : \alpha(x)\}$$

Note that the empty set can be specified in this way by giving a condition $\alpha(x)$ in GPLIS:

$$\emptyset = \{x : \neg x = x\}$$

Because $\forall x x = x$ is logically true, no object satisfies the condition $\neg x = x$; hence, the set of all and only the objects that satisfy this condition is the empty set. Assuming that sets S and T have been specified in this way—that is, that we have introduced "S" as a name for a certain set specified by some condition and "T" as a name for a certain set specified by some condition—the sets S', $S \cup T$, $S \cap T$, and so on can also be specified in this way. That is precisely how we did specify them in §16.1.3: with conditions stated using only logical symbols of GPLI and the new symbol $\in$ (and the names "S" and "T").

Let's return to the precisified version of the intuitive thought. It is known as the principle of *unrestricted comprehension* (or "unlimited comprehension"):

For any wff $\alpha(x)$ in GPLIS containing one or more free occurrences of x, there exists a set:

$$\{x : \alpha(x)\}$$

The term *naïve set theory* is often used for the theory of sets that takes extensionality and unrestricted comprehension as its basic principles. Unrestricted comprehension determines which sets exist; extensionality determines when sets x and y are one and the same set. Frege [1964, p. 105] took as an axiom (Basic Law V) in his later formal system a principle that implies both a version of unrestricted comprehension and extensionality. However—as Russell [1902] pointed out to Frege in a now-famous letter, we can derive a contradiction from the principle of unrestricted comprehension. Let $\alpha(x)$ be the formula $\neg x \in x$. Then the principle yields a set $\{x : \neg x \in x\}$. Call this set R (the Russell set). By pure logic, either $R \in R$ or $\neg R \in R$. Suppose the former: then $\neg R \in R$ (because the condition required for R to be in R is $\neg R \in R$). Suppose the latter: then it is not the case that $\neg R \in R$ (again because the condition required for R to be in R is $\neg R \in R$, so if R is not in R, it must be that the condition is not satisfied); that is, $R \in R$. Thus, we have $R \in R \vee \neg R \in R$, $R \in R \rightarrow \neg R \in R$ and $\neg R \in R \rightarrow R \in R$. From these, the contradiction $R \in R \wedge \neg R \in R$ follows by pure logic. We have derived a contradiction ($R \in R \wedge \neg R \in R$) from the principle of unrestricted comprehension. This is *Russell's Paradox.*

We therefore need a new theory about which sets exist: unrestricted comprehension will not do. A common picture nowadays concerning which sets exist is the *iterative conception of set.* In this view, sets are built up in stages. A set S can only be built at stage x if all members of S already exist as of stage x. In particular, a set that contains sets as members can only be built at stage x if these member sets were built at some stage prior to x.

We start building sets at stage 0. At this stage—as we have not yet built any sets—all we have available to put in the sets we are building are objects that are not sets; these are called *urelements*. There may be no urelements; as we shall see, we can still build plenty of sets in this case. At stage 0 we can always build the empty set. If there are no urelements, this is the only set we can build. If there is one urelement, a, we can build the sets $\emptyset$ and $\{a\}$. If there are two urelements, a and b, the possible sets are $\emptyset$, $\{a\}$, $\{b\}$, and $\{a, b\}$; and so on if there are more urelements.

At stage 1, we can build any set containing urelements or sets built at stage 0, that is, any set whose members are already available at the beginning of stage 1. If there are no urelements, we can build $\emptyset$ and $\{\emptyset\}$. (Note that $\emptyset$ was already built at stage 0. At every stage, we can always build again everything built at any earlier stage. In general, when talking about the stage at which a set is formed, we mean the earliest stage at which it is formed.) If there is one urelement, a, then at stage 1 we can build the following eight sets:

$$\begin{array}{c} \emptyset \\ \{a\} \qquad \{\emptyset\} \qquad \{\{a\}\} \\ \{a, \emptyset\} \qquad \{a, \{a\}\} \qquad \{\emptyset, \{a\}\} \\ \{a, \emptyset, \{a\}\} \end{array}$$

(Two of these—$\emptyset$ and $\{a\}$—were already built at stage 0.) If there are more urelements, we can build even more sets at this stage.

At stage 2, we can build any set containing urelements, sets built at stage 0, or sets built at stage 1, that is, any set whose members are already available at the beginning of stage 2. If there are no urelements, we can build the following four sets:

$$\begin{array}{c} \emptyset \\ \{\emptyset\} \qquad \{\{\emptyset\}\} \\ \{\emptyset, \{\emptyset\}\} \end{array}$$

(The empty set $\emptyset$ was already built at stage 0 and at stage 1; $\{\emptyset\}$ was already built at stage 1.) If there is one urelement, a, then at stage 2 we have nine objects available to put into sets: a, and the eight sets built at stage 1 (the two sets built at stage 0 were also built at stage 1, so we do not count them again). Thus, we can build $2^9 = 512$ sets (too many to show here). If we have more urelements, we can build even more.

The progression of stages never stops: indeed, it extends to *transfinite* stages. Thus, it is not just that there is a stage n for every finite n: after all these finite stages (infinitely many of them), there is another stage, stage ω. At this stage, we form sets whose members may be any urelement (if there are any), or any set formed at any earlier stage $(1, 2, 3, \ldots)$. Next we have a stage $\omega + 1$, at which we form sets whose members may be any urelement (if there are any), or any set formed at any earlier stage $(1, 2, 3, \ldots, \omega)$; and so on, through stages $\omega + 2$, $\omega + 3$, . . . , $\omega + \omega (= \omega.2)$, $\omega.2 + 1$, $\omega.2 + 2$, $\omega.2 + 3$, . . . , $\omega.2 + \omega (= \omega.3)$, . . . , $\omega.\omega$, ...

Sets built up in this way from no urelements are called *pure* sets. They can be arranged into a hierarchy—known as the *cumulative* or *iterative* hierarchy—according to the stage at which they are (first) formed. All the usual objects considered in mathematics can be identified with sets in the cumulative hierarchy. For example, the natural numbers 0, 1, 2, . . . can be identified with the sets $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, . . . (note that each set in the sequence contains all the earlier sets in the sequence). At the same time, certain problematic sets are not built at any stage—and so they do not exist at all, in this conception. For example, there is no Russell set. For note that no set in the cumulative hierarchy is a member of itself: a set S can only have as members things that have already been formed prior to the stage at which S is formed; so for S to contain itself, S would have to be formed at some stage prior to the stage at which S is formed—which is impossible. Thus, the set of all sets that are not members of themselves would simply be the set of all sets in the cumulative hierarchy. But there is no such set. For if there were, it would have to be formed at some stage—and then it would not contain the sets formed at subsequent stages (remember, the progression of stages never ends).

The iterative conception thus provides a theory about what sets exist that yields enough sets for mathematics and promises to avoid contradictions, such as Russell's paradox. Of course, the theory—as we have presented it here—is not precise. Greater precision may be attained by formulating axioms that are true in the cumulative hierarchy and then working directly from the axioms; this is known as *axiomatic set theory*.[8]

16.2 Ordered Pairs and Ordered *n*-tuples

Roughly speaking, an *ordered pair* consists of two objects, given in a particular order: one first, the other second. The ordered pair consisting of Alice first and Bob second is represented as ⟨Alice, Bob⟩ or (Alice, Bob). An *ordered triple* consists of three objects, given in a particular order. The ordered triple consisting of Alice first, Bob second and Carol third is represented as ⟨Alice, Bob, Carol⟩ or (Alice, Bob, Carol). In general, an *ordered n-tuple* (or just an n-tuple, for short) consists of n objects in a particular order. The ordered n-tuple consisting of Alice first, Bob second, . . . , and Carol in nth position is represented as ⟨Alice, Bob, . . . , Carol⟩ or (Alice, Bob, . . . , Carol). "Ordered pair" is then just another term for an ordered 2-tuple, and "ordered triple" is another term for an ordered 3-tuple.

I said "roughly speaking" because in fact an ordered pair does not have to comprise two different objects, an ordered triple does not have to comprise three different objects, and so on. For example, ⟨1, 1⟩, ⟨Alice, Alice⟩, and

$\langle$Bob, Bob$\rangle$ are perfectly good ordered pairs. Here we have just one object (in each case) that occupies both positions in the pair. Thus, we should think of an ordered pair not as "two objects" given in a certain order, but as an abstract ranking or ordering with two positions, first and second: a stipulation of a first object and a second object (which may or may not be the same object). There is, in general, no reason why Alice (or any other individual) should not be ranked first *and* second. For example, suppose that some children are working out an ordering of who gets to go on the swing. If Alice has been sick in bed for a week and has just rejoined the group, the children might deem that not only should she have first go, she should have two goes in a row. Thus, she occupies positions one and two in the ordering. The idea is not that she is standing behind herself in a queue: that she is both first and second in line. That is impossible. Rather, the ordering—in which she occupies both first and second position—is an abstract thing.

The same point applies to ordered triples, and indeed to ordered n-tuples in general. Thus, the following are all perfectly good ordered triples, and they are all different triples:

$$\langle 1, 2, 3\rangle \quad \langle 1, 2, 1\rangle \quad \langle 1, 2, 2\rangle \quad \langle 3, 2, 3\rangle \quad \langle 1, 1, 1\rangle \quad \langle 3, 3, 3\rangle \quad \langle 2, 2, 1\rangle$$

In general, there must be at least one, and at most n, distinct objects in an ordered n-tuple. At one extreme we have the same object occupying all positions; at the other extreme we have different objects in every position.

We saw that for sets, the order in which one writes the members is irrelevant; for example, $\{1, 2\} = \{2, 1\}$, and $\{1, 2, 3\} = \{2, 1, 3\}$. For ordered n-tuples, this is not the case: $\langle 1, 2\rangle \neq \langle 2, 1\rangle$ and $\langle 1, 2, 3\rangle \neq \langle 2, 1, 3\rangle$. We also saw that $\{1, 1\}$ is just another way of writing $\{1\}$, $\{1, 1, 2, 2\}$ is just another way of writing $\{1, 2\}$, and so on. For ordered n-tuples, this is not the case: $\langle 1, 1\rangle \neq \langle 1\rangle$ and $\langle 1, 1, 2, 2\rangle \neq \langle 1, 2\rangle$. The n-tuple $\langle 1, 1\rangle$ is an ordered pair with 1 in both positions (first and second); $\langle 1\rangle$ is an ordered 1-tuple with 1 in its first (and only) position. Similar remarks apply to $\langle 1, 1, 2, 2\rangle$ and $\langle 1, 2\rangle$.

For any object and any set, there are only two possibilities: the object is either in the set, or it isn't. So, a is in the set $\{a, c\}$, b isn't, and c is. If we write something like $\{a, a, c\}$, we have just written the same set as before—the one that has a and c in it, and nothing else—only in a more long-winded way. We can write a twice, but a can't be in the set twice: it is either in, or it isn't—there are no different grades or ways of being in a set. For an object and an ordered n-tuple, however, the question is not simply whether the object is in the n-tuple. The question is: where in the n-tuple is the object? The following are therefore three different n-tuples:

$$\langle a, c\rangle \quad \langle a, a\rangle \quad \langle a, a, c\rangle$$

The first is a 2-tuple (i.e., an ordered pair) in which a appears in first position, c appears in second position, and no other object appears. The second is also a 2-tuple, but this time, a appears twice—in first and second positions—and no other object appears. The third is a 3-tuple (i.e., an ordered triple), in which a appears in first and second positions, c appears in third position, and no other object appears.

For any sets S and T, their *Cartesian product* $S \times T$ is the set of all ordered pairs whose first member is an element of S and whose second member is an element of T. For example, if $S = \{1, 2\}$ and $T = \{3, 4\}$, then

$$S \times T = \{\langle 1, 3\rangle, \langle 1, 4\rangle, \langle 2, 3\rangle, \langle 2, 4\rangle\}$$

Where S and T are the same set, the Cartesian product $S \times S$ is denoted S^2. For example, if $S = \{1, 2, 3\}$, then

$$S^2 = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle, \langle 2, 3\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle, \langle 3, 3\rangle\}$$

The set of all ordered triples of elements of S is denoted S^3, and in general the set of all ordered n-tuples of elements of S is denoted S^n. For example, if $S = \{1, 2\}$, then

$$S^3 = \{\langle 1, 1, 1\rangle, \langle 1, 1, 2\rangle, \langle 1, 2, 1\rangle, \langle 1, 2, 2\rangle, \langle 2, 1, 1\rangle, \langle 2, 1, 2\rangle, \langle 2, 2, 1\rangle, \langle 2, 2, 2\rangle\}$$

Note that the rows in the matrix of a truth table for a proposition (or collection of propositions) containing n basic propositions are precisely the ordered n-tuples in $\{T, F\}^n$, where $\{T, F\}$ is the set of truth values. (In each row of the matrix, the first entry is the truth value of the first basic proposition, the second entry is the truth value of the second basic proposition, . . . , and the final—nth—entry is the truth value of the nth basic proposition. The rows cover all possible assignments of values to these propositions—that is, all possible n-tuples of values.)

16.2.1 Reduction to Sets

Ordered pairs do not have to be thought of as a new kind of primitive entity: they can be identified with sets of a certain sort. This can be done in various ways; the now-standard approach is due to Kuratowski [1921, 171]:[9]

$$\langle a, b\rangle = \{\{a, b\}, \{a\}\}$$

The essential thing about an ordered pair is that it specifies which object comes first and which comes second. In other words, the identity condition for ordered pairs is: if we have an ordered pair x and an ordered pair y, they are one and the same ordered pair iff x's first object is the same as y's first object and x's second object is the same as y's second object. In symbols:

$$\langle x, y\rangle = \langle z, w\rangle \leftrightarrow (x = z \wedge y = w)$$

The key aspect of a reduction of ordered pairs to sets is that this identity condition should then follow from the identity condition (i.e., extensionality) for the sets to which $\langle x, y\rangle$ and $\langle z, w\rangle$ are reduced (i.e., here, $\{\{x, y\}, \{x\}\}$ and $\{\{z, w\}, \{z\}\}$). That is, it should follow from extensionality that:

$$\{\{x, y\}, \{x\}\} = \{\{z, w\}, \{z\}\} \leftrightarrow (x = z \wedge y = w)$$

The right-to-left direction holds trivially. For the left-to-right direction, suppose that $\{\{x, y\}, \{x\}\} = \{\{z, w\}, \{z\}\}$ (call this identity A). We want to show that $x = z$ and $y = w$. There are two cases to consider:

(i) $x = y$. In this case, $\{x, y\} = \{x, x\} = \{x\}$, so $\{\{x, y\}, \{x\}\} = \{\{x\}, \{x\}\} = \{\{x\}\}$. So A becomes $\{\{x\}\} = \{\{z, w\}, \{z\}\}$, from which it follows, by extensionality, that both $\{z, w\}$ and $\{z\}$ are in $\{\{x\}\}$; that is, $\{z, w\} = \{x\}$, and $\{z\} = \{x\}$. By extensionality the former yields $z = x$ (and so $x = z$) and $w = x$; from $x = y$ and $w = x$ we get $y = w$.

(ii) $x \neq y$. Hence, $\{x, y\}$ is a two-membered set, so $\{x, y\} \neq \{x\}$ (as $\{x\}$ is a one-membered set, and by extensionality a two-membered set cannot be identical to a one-membered set), and so $\{\{x, y\}, \{x\}\}$ is a two-membered set. Hence, given A, $\{\{z, w\}, \{z\}\}$ must also have two members, so $z \neq w$. Furthermore, one of the members of $\{\{z, w\}, \{z\}\}$ must be $\{x, y\}$, and the other must be $\{x\}$. As $\{x, y\}$ and $\{z, w\}$ are both two-membered and $\{x\}$ and $\{z\}$ are one-membered, (a) $\{x, y\} = \{z, w\}$, and (b) $\{x\} = \{z\}$. From (b), $x = z$. From (a), and $x \neq y$ and $z \neq w$ and $x = z$, it follows that $y = w$.

Other reductions would also work—for example, we could say that $\langle a, b\rangle = \{\{a, b\}, \{b\}\}$. Not anything would work, however—for example, if we said that $\langle a, b\rangle = \{\{a\}, \{b\}\}$ then it would turn out that $\langle a, b\rangle = \langle b, a\rangle$ (even when $a \neq b$).

What about ordered n-tuples, where n is a number other than 2? An ordered 1-tuple $\langle x\rangle$ can simply be identified with the set $\{x\}$. An ordered triple $\langle x, y, z\rangle$ can be identified with the ordered pair $\langle\langle x, y\rangle, z\rangle$. Note that the first member of this ordered pair is itself an ordered pair. An ordered 4-tuple $\langle x, y, z, w\rangle$ can then be identified with the ordered pair $\langle\langle x, y, z\rangle, w\rangle$. Note that the first member of this ordered pair is an ordered triple. Given that we have seen how to reduce an ordered triple to ordered pairs, this representation shows how to reduce an ordered 4-tuple to ordered pairs. In general, we can reduce the ordered $(n + 1)$-tuple $\langle x_1, \ldots, x_n, y\rangle$ to the ordered pair $\langle\langle x_1, \ldots, x_n\rangle, y\rangle$, and thus all ordered n-tuples $(n > 2)$ may ultimately be reduced to ordered pairs—which, as we have seen, may be reduced to sets.[10]

A second approach to ordered n-tuples, for $n > 2$, is to view an ordered n-tuple as a *sequence* of length n—in the precise sense of "sequence" introduced in §16.5. As we shall see, a sequence in this sense is a certain sort of function, and a function may be seen as a certain sort of set of ordered pairs. Thus, it would be circular to identify ordered pairs with sequences of length 2. However, once we identify ordered pairs with sets in the way discussed above, we are then free to identify ordered n-tuples, for $n > 2$, with sequences of length n.

16.3 Relations

An n-place relation is a condition that an n-tuple of objects may or may not satisfy; thus, we think of it as a set of n-tuples. For example, consider the relation "x is a brother of y." Let's say Bill is a brother of Ben, and vice versa; Bill is a brother of Carol, but not vice versa; and Ben is a brother of Carol, but not vice versa. Then we can think of this relation as the following set of ordered pairs:

$$\{\langle \text{Bill, Ben} \rangle, \langle \text{Ben, Bill} \rangle, \langle \text{Bill, Carol} \rangle, \langle \text{Ben, Carol} \rangle\}$$

A 2-place relation is also called a *binary* relation; a 3-place relation is also called a *ternary* relation.

Often we want to be quite specific about the sets from which the elements of the ordered n-tuples in a relation (a set of n-tuples) are drawn. We say that a binary relation *from* a set S *to* a set T is a subset of $S \times T$, that is, a set of ordered pairs whose first elements are in S and whose second elements are in T. Where S and T are the same set, a binary relation from S to T—that is, from S to itself—is also called a binary relation *on* S. A binary relation on S is a subset of S^2. Similarly, a ternary relation on S is a subset of S^3 (i.e., a set of ordered triples of elements of S), and in general an n-place relation on S is a subset of S^n.

There are various properties that a binary relation R on S may have. For example,

Reflexivity: for every x in S, $\langle x, x \rangle \in R$. That is, the relation holds between every object x and itself.

Irreflexivity: for every x in S, $\langle x, x \rangle \notin R$. That is, the relation holds between no object x and itself.

Transitivity: for every x, y, and z in S, if $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in R$, then $\langle x, z \rangle \in R$. That is, if the relation holds between x and y and between y and z, then it holds between x and z.

Symmetry: for every x and y in S, if $\langle x, y \rangle \in R$, then $\langle y, x \rangle \in R$. That is, if the relation holds between x and y in one order, then it holds between them in the other order as well.

Antisymmetry: for every x and y in S, if $\langle x, y \rangle \in R$ and $\langle y, x \rangle \in R$, then $x = y$. That is, the only case in which the relation holds between x and y in both directions is the case where x and y are one and the same object.[11]

Asymmetry: for every x and y in S, if $\langle x, y\rangle \in R$ then $\langle y, x\rangle \notin R$. That is, if the relation holds between x and y in one direction, then it does not hold in the other direction.[12]

Connectedness: for every x and y in S such that $x \neq y$, either $\langle x, y\rangle \in R$ or $\langle y, x\rangle \in R$ (or both). That is, for any distinct objects x and y, the relation holds between them in at least one direction.[13]

Relations having certain groups of these properties are important in certain contexts and hence have been given special names. Three examples of such relations are equivalence relations, partial orders, and linear orders.

R is an *equivalence relation* if it is reflexive, symmetric, and transitive. Note that if R is an equivalence relation, then it divides S into subsets—*equivalence classes*—with the following features: every member of S is in exactly one equivalence class (i.e., the equivalence classes are nonoverlapping and between them they cover all of S; i.e., they constitute a *partition* of S); for any x and y in S (including the case $x = y$), $\langle x, y\rangle \in R$ iff x and y are in the same equivalence class. The identity relation on any set S is an equivalence relation; each equivalence class contains exactly one object. The relation of logical equivalence on the set of wffs of PL is an equivalence relation; if an equivalence class contains a wff α, then it also contains all and only the wffs that are logically equivalent to α.

R is a *partial order* if it is reflexive, transitive, and antisymmetric. It is a *strict* partial order if it is irreflexive and transitive (it follows that it must also be asymmetric). R is a (strict) *linear order* if it is a (strict) partial order that is also connected. The relation $\leq$ on the natural numbers is a linear order (and hence also a partial order); the relation $<$ on the natural numbers is a strict linear order (and hence also a strict partial order). For any set S, the relation $\subseteq$ on $\wp S$ is a partial order (but not in general a linear order);[14] the relation $\subsetneq$ on $\wp S$ is a strict partial order (but not in general a strict linear order). Note that given any partial order, if we remove all pairs $\langle x, x\rangle$, the result will be a *strict* partial order; given any strict partial order, if we *add* all pairs $\langle x, x\rangle$ (for all x in S), the result will be a partial order. An analogous result holds for linear orders and strict linear orders.

16.4 Functions

A *function* (aka map, mapping, operation) f from a set S to a set T, written:[15]

$$f : S \to T$$

assigns particular objects in T to objects in S. S is called the *domain* of the function and T the *codomain*. The essential feature of a function is that it never assigns more than one object in T to any given object in S. If x is a member of

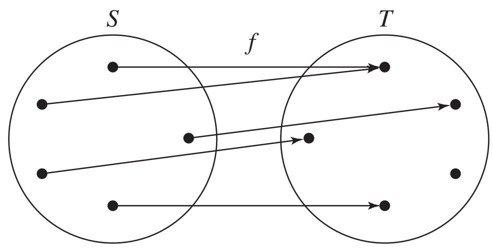


Figure 16.2. Picturing a function as a collection of arrows.

S, $f(x)$ is the object in T that the function f assigns to x. We say that $f(x)$ is the *value* or *output* of the function f for the *argument* or *input* x, or it is the value *at* x; we also say x is *sent to* $f(x)$, that $f(x)$ is *hit by* x, or that $f(x)$ is the *image* of x under f. Note that S and T may be the same set. In this case we call a function from S to T—that is, from S to itself—a function *on* S.

A function $f : S \to T$ is commonly identified with the set of ordered pairs $\langle x, f(x)\rangle$, where x is an object in S that is sent to some object in T by f, and $f(x)$ is the object in T to which x is sent. For example, consider the *successor* function on the set of natural numbers, which, given a number as input, yields as output the *next* number in the sequence of natural numbers. Represented as a set of ordered pairs, it is:

$$\{\langle 0, 1\rangle, \langle 1, 2\rangle, \langle 2, 3\rangle, \ldots\}$$

The crucial feature of a function—that it never assigns more than one object in T to any given object in S—emerges here as the requirement that no element of S appears more than once as the first element of an ordered pair in the set.

Another useful way to picture a function $f : S \to T$ is as a collection of arrows pointing from objects x in S to objects $f(x)$ in T (Figure 16.2). Binary relations can also usefully be pictured as collections of arrows. In this depiction, functions are distinguished from relations in general by the requirement on functions that no object has more than one arrow departing from it.

As with relations, there are various properties which a function $f : S \to T$ may have. For example:

A function $f : S \to T$ is said to be *total* if it satisfies the condition that every member of S is sent to some member of T. A function that is not total is called *partial*. Such a function assigns nothing to some member(s) of S. In the representation of a function as a set of ordered pairs, to say that $x \in S$ is assigned no value by the partial function $f : S \to T$ means that x does not appear as the first element of any ordered pair in the set; in the representation of a function as a collection of arrows, it means that x has no arrow leading from it.[16]

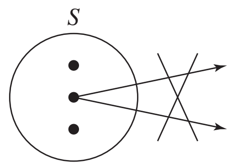
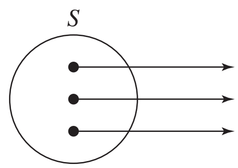
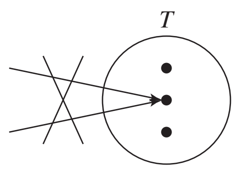
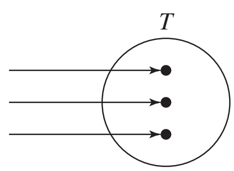
	function (we never see two arrows leaving the same object)
	total (every object has an arrow leaving it)
	one-one (we never see two arrows hitting the same object)
	onto (every object gets hit by an arrow)

Figure 16.3. Kinds of function from S to T.

A function $f : S \to T$ is said to be *onto* (aka surjective, a surjection) if it satisfies the condition that every member of T is hit at least once; *one-one* (aka one-to-one, into, injective, an injection) if no member of T is hit more than once; and a *correspondence* (aka bijective, a bijection) if it is total, onto, and one-one. (See Figure 16.3. Note that the top property in the table, unlike the three below it, is a sine qua non for functions: if a subset of $S \times T$ does not possess this property, then it is not a function. Also, do not be misled by the picture of an onto function: a function can be onto without being one-one.)

Note that if (as discussed above) we identify a function f with the set of all ordered pairs $\langle x, y\rangle$ such that $f(x) = y$, then the following identity condition for functions holds: $f = g$ iff f assigns values to all and only the objects to which g assigns values, and for all such objects x, $f(x) = g(x)$. This condition follows from the identity condition for sets (extensionality) together with the identity condition for ordered pairs given in §16.2.1 (which itself follows from extensionality, if we identify ordered pairs with sets in the way discussed in §16.2.1).

So far we have considered functions that take a single object as argument and assign to it an object as value. What about functions, such as addition or multiplication, that take two objects as arguments and assign to them an object as value? (These are called *binary* functions.) In general, what about functions that take three, four, or in general n objects as arguments? One common way of conceiving of such functions, which brings them within the framework articulated above for one-place functions, is to conceive of an apparently n-place function from S to T as a (one-place) function from S^n to T. That is, it is a one-place function that takes as input an n-tuple of objects. (An n-tuple, like a set, is considered to be a single object.) So, for example, the addition function, which we normally think of as taking two numbers as input, may be thought of as taking a single input: an ordered pair of numbers.

16.4.1 Operations on Functions

Given a function $f : S \to T$, we can invert the function: switch the first and second members of each ordered pair (make each arrow point the opposite way). If the result of this process is a function (from T to S), this resulting function is called the *inverse* function of f and is denoted by f^{-1}. If the result is not a function, we say f^{-1} does not exist. (Of course, the set of switched-around ordered pairs always exists: it's just that it might not be a function: f^{-1} names the inverse function, if it exists.) It's not too hard to see that the condition required for the set of switched-around ordered pairs to be a function is that f is one-one. Furthermore, provided f^{-1} exists:

- f^{-1} is one-one (because f is a function).
- If f is total, then f^{-1} is onto.
- If f is onto, then f^{-1} is total.

To see why these statements are true, it is helpful to recall Figure 16.3.

Given a function $f : S \to T$ and a function $g : T \to U$, the *composite function* $g \circ f$ (read as "g after f") from S to U is defined thus: for every x in S, $(g \circ f)(x) = g(f(x))$. The idea here is that we first apply f to the input x (a member of the set S) and then apply g to the result (i.e., to the output of f for input x, which is a member of T). In terms of arrows, the composite function is found by taking each f arrow from an object x in S to an object y in T and extending it so that it hits whatever object z in U the g arrow from y hits. (If there is no g arrow from y, then in the composite function there is no arrow from x.) Think through why the following must be true:

- If f and g are both total, so is $g \circ f$.
- If f and g are both onto, so is $g \circ f$.

- If f and g are both one-one, so is $g \circ f$.
- If f and g are both bijections, so is $g \circ f$.

16.4.2 Characteristic Function of a Set

Given a subset S of a background set U, the *characteristic* (or indicator) function of S is a total function $\mathbf{I}_S : U \to \{0, 1\}$ defined as follows. For all x in U:

$$\mathbf{I}_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

We can think of the characteristic function as answering "yes" (1) or "no" (0), for every object in U, to the question whether that object is in S.

Instead of the set $\{0, 1\}$, we might take the set {T, F} of truth values as the codomain of the characteristic function (with T being "yes" and F being "no"). Conversely, it is also common to take $\{0, 1\}$ as the set of truth values: that is, to use 1 everywhere we have used T and 0 everywhere we have used F (e.g., in truth tables).

Think through why the following are true for any subsets S and T of a background set U.

$$\text{For every } x \in U, \qquad \begin{aligned} \mathbf{I}_{S \cup T}(x) &= \max\{\mathbf{I}_S(x), \mathbf{I}_T(x)\} \\ &= \mathbf{I}_S(x) + \mathbf{I}_T(x) - [\mathbf{I}_S(x) \times \mathbf{I}_T(x)] \end{aligned}$$

Note here that where x and y are numbers, $\max\{x, y\}$ is the greater of x and y; if $x = y$, then $\max\{x, y\} = x$.

$$\text{For every } x \in U, \qquad \begin{aligned} \mathbf{I}_{S \cap T}(x) &= \min\{\mathbf{I}_S(x), \mathbf{I}_T(x)\} \\ &= \mathbf{I}_S(x) \times \mathbf{I}_T(x) \end{aligned}$$

Note here that where x and y are numbers, $\min\{x, y\}$ is the lesser of x and y; if $x = y$, then $\min\{x, y\} = x$.

$$\begin{array}{ll} \text{For every } x \in U, & \mathbf{I}_{S'}(x) = 1 - \mathbf{I}_S(x) \\ S \subseteq T \text{ iff for every } x \in U, & \mathbf{I}_S(x) \leq \mathbf{I}_T(x) \\ \text{For every } \langle x, y \rangle \in U^2, & \mathbf{I}_{S \times T}(\langle x, y \rangle) = \mathbf{I}_S(x) \times \mathbf{I}_T(y) \end{array}$$

In relation to the last of these facts, note that $S \times T$ is a subset of U^2, so the characteristic function of $S \times T$ is a function from U^2 to $\{0, 1\}$.

Chapter 16: Set Theory

1. "Unter einer 'Mannigfaltigkeit' oder 'Menge' verstehe ich nämlich allgemein jedes Viele, welches sich als Eines denken läßt" [Cantor, 1932, 204].

2. This usage is not the same as—but clearly related to—the usage of "extension" to mean the value of a predicate.

3. Some works use the symbol $\subset$ to indicate proper subset (and use $\subseteq$ in the way we do here), but others use $\subset$ to mean exactly what we mean by $\subseteq$.

4. Sometimes $\bar{S}$ or S^c is written instead of S'.

5. Sometimes $S - T$ is written instead of $S \setminus T$; sometimes the former notation is restricted to contexts where $T \subseteq S$.

6. Consider the definitions of union and intersection. For any x, $x \in S \cup T$ iff $x \in S \vee x \in T$, and $x \in T \cup S$ iff $x \in T \vee x \in S$. But $\alpha \vee \beta$ and $\beta \vee \alpha$ are equivalent, so $x \in S \cup T$ iff $x \in T \cup S$—hence, $S \cup T = T \cup S$. Likewise, for any x, $x \in S \cap T$ iff $x \in S \wedge x \in T$, and $x \in T \cap S$ iff $x \in T \wedge x \in S$. But $\alpha \wedge \beta$ and $\beta \wedge \alpha$ are equivalent, so $x \in S \cap T$ iff $x \in T \cap S$—hence, $S \cap T = T \cap S$.

7. Consider the definition of set-theoretic difference. For any x, $x \in S \setminus T$ iff $x \in S \wedge \neg x \in T$, and $x \in T \setminus S$ iff $x \in T \wedge \neg x \in S$. In general, $\alpha \wedge \neg\beta$ is not equivalent

to $\beta \wedge \neg\alpha$, so it is not in general the case that $x \in S \setminus T$ iff $x \in T \setminus S$—hence, in general $S \setminus T \neq T \setminus S$. However, when α and β are equivalent, $\alpha \wedge \neg\beta$ *is* equivalent to $\beta \wedge \neg\alpha$: both are equivalent to the contradiction $\alpha \wedge \neg\alpha$. This corresponds to the fact that when $S = T$ (i.e., $x \in S$ iff $x \in T$), $S \setminus T = T \setminus S = \emptyset$.

8. There are various systems of axiomatic set theory. Not all of them feature axioms that are true in the cumulative hierarchy. However, what is generally considered to be the standard set of axioms for set theory—ZFC (Zermelo-Fraenkel set theory with the axiom of Choice)—does: see Boolos [1971], Shoenfield [1977], and Devlin [1993, pp. 29–65].

9. For the history of early reductions of ordered pairs to sets, see van Heijenoort [1967, 224].

10. Of course, there are other equally good options; for example, we could reduce the ordered $(n+1)$-tuple $\langle x, y_1, \ldots, y_n\rangle$ to the ordered pair $\langle x, \langle y_1, \ldots, y_n\rangle\rangle$.

11. In this case, "both" directions are really one and the same. If $x = y$, then R holding between x and y in that order, and R holding between x and y in the other order, are just the same thing: R holding between x and x. While there are two ordered pairs containing both the objects 1 and 2 (i.e., $\langle 1, 2\rangle$ and $\langle 2, 1\rangle$), there is just one ordered pair containing the object 1 (i.e., $\langle 1, 1\rangle$). This follows from the identity condition for ordered pairs discussed in §16.2.1.

12. Given what we said in n. 11, evidently a relation that is asymmetric must also be irreflexive.

13. This condition says nothing about the case $x = y$: all the different possibilities—$\langle x, x\rangle \in R$ for all x (reflexivity), $\langle x, x\rangle \in R$ for no x (irreflexivity), $\langle x, x\rangle \in R$ for some but not all x (neither reflexive nor irreflexive)—are compatible with connectedness.

14. That is, a partial order on $\wp S$, not on S.

15. This use of the arrow symbol has nothing to do with our use of this symbol for the conditional: these are simply two different uses of the same symbol. This phenomenon—where the same term or symbol means different things in different contexts—is common in logic and mathematics.

16. Readers coming from certain backgrounds might not be used to thinking of partial functions as functions at all. They might take a "function" from S to T to mean a subset of $S \times T$, where every element of S appears *exactly* once as the first element of an ordered pair in the set. Here—and this is standard in logic—we take a function from S to T to be a subset of $S \times T$, where every element of S appears *at most* once as the first element of an ordered pair in the set. If, in addition, every element of S appears once as the first element of an ordered pair in the set, it is a total function; if not, it is a partial function—but we still count it as a function. Partial functions arise naturally at many points in logic.

17. It is also useful in some contexts to countenance the *empty* sequence of members of S: the sequence with no entries, which has length zero (no positions). In the sense in which we have just defined finite and infinite sequences, the empty sequence is neither a finite sequence nor an infinite sequence (it is not a total function from any initial segment—as we defined "initial segment"—of $\mathbb{Z}^+$ to S, nor is it a total function from $\mathbb{Z}^+$ to S). We shall not consider the empty sequence further here, but note that if we did wish to include it when we spoke of "all sequences of members of S" (which, as mentioned, is useful in some contexts), we would need to alter or augment our existing definition of a sequence to include it.

18. If we chose instead to take the other fact about characteristic functions of intersections stated in §16.4.2 as our starting point—that is, $\mathbf{I}_{S\cap T}(x) = \mathbf{I}_S(x) \times \mathbf{I}_T(x)$—then we would obtain a very different notion of intersection of multisets: one according to which the intersection of a multiset that contains a twice and a multiset that contains a five times contains a ten times. Likewise, in the case of union, it is more natural to take the first fact stated in §16.4.2 as our starting point—that is, $\mathbf{I}_{S\cup T}(x) = \max\{\mathbf{I}_S(x), \mathbf{I}_T(x)\}$.

19. The notion of complement of multisets is a bit more subtle. For a start, the fact about characteristic functions of complements stated in §16.4.2—that is, $\mathbf{I}_{S'}(x) = 1 - \mathbf{I}_S(x)$—is not much help when $\mathbf{I}$ may take values greater than 1. For a discussion, see Hickman [1980]. For more details about multisets, see, for example, Syropoulos [2001].

20. In the case of PL, the corresponding clause says "any basic proposition is a wff." This clause tells us that for any basic proposition x in S, the length-1 sequence $\langle x\rangle$, which has x in its only position, is in W.

21. Again, because concatenation is defined on sequences, we cannot concatenate α directly with the symbols $\forall$ and $\underline{x}$: we have to concatenate α with the length-1 sequences $\langle\forall\rangle$ and $\langle\underline{x}\rangle$.