3.5 Orderings

An *order* is a binary relation which is transitive and in addition either (i) reflexive and antisymmetric or else (ii) irreflexive and asymmetric. The former are *weak* orders; the latter are *strict* (or *strong*).

To illustrate, let $A = \{a, b, c, d\}$. The following are all weak orders in A:

(3–21) $R_1 = \{\langle a, b\rangle, \langle a, c\rangle, \langle a, d\rangle, \langle b, c\rangle, \langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle d, d\rangle\}$
$R_2 = \{\langle b, a\rangle, \langle b, b\rangle, \langle a, a\rangle, \langle c, c\rangle, \langle d, d\rangle, \langle c, b\rangle, \langle c, a\rangle\}$
$R_3 = \{\langle d, c\rangle, \langle d, b\rangle, \langle d, a\rangle, \langle c, b\rangle, \langle c, a\rangle, \langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle d, d\rangle, \langle b, a\rangle\}$

These are represented in Figure 3-3 as relational diagrams, from which it can be verified that each is indeed reflexive, antisymmetric, and transitive

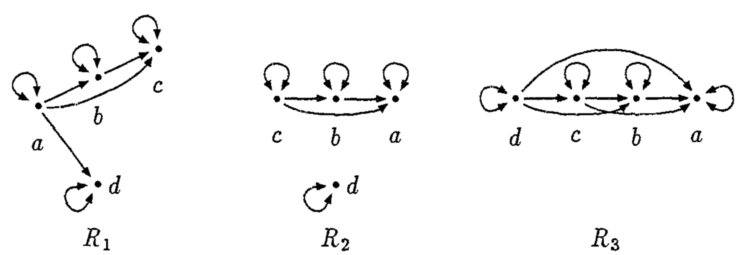

Figure 3–3:
Diagrams of the weak orders in (3-21).

To these weak orders there correspond the strict orders S_1, S_2 and S_3, respectively:

(3–22) $S_1 = \{\langle a, b\rangle, \langle a, c\rangle, \langle a, d\rangle, \langle b, c\rangle\}$
$S_2 = \{\langle b, a\rangle, \langle c, b\rangle, \langle c, a\rangle\}$
$S_3 = \{\langle d, c\rangle, \langle d, b\rangle, \langle d, a\rangle, \langle c, b\rangle, \langle c, a\rangle, \langle b, a\rangle\}$

These can be gotten from the weak orders by removing all the ordered pairs of the form $\langle x, x\rangle$. Conversely, one can make a strict order into a weak order by adding the pairs of the form $\langle x, x\rangle$ for every x in A.

As another example of an order, consider any collection of sets C and a relation R in C defined by $R = \{\langle X, Y\rangle \mid X \subseteq Y\}$ We have already noted in effect (Chapter 1, section 4) that the subset relation is transitive and reflexive. It is also antisymmetric, since for any sets X and Y, if $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$ (this will be proved in Chapter 7). The corresponding strict order is the 'proper subset of' relation in C.

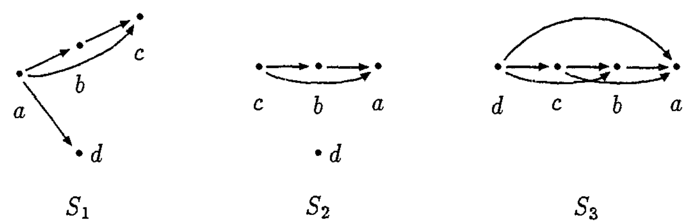

Figure 3–4:
Diagrams of the strict orders in (3-22).

Further, we saw in Example (3-13) that the relation R 'greater than' in the set of positive integers is irreflexive, asymmetric and transitive. It is therefore a strict order. (*Problem*: What relation defines the corresponding weak order?)

Some terminology: if R is an order, either weak or strict, and $\langle x, y \rangle \in R$, we say that *x precedes y, x is a predecessor of y, y succeeds* (or *follows*) x, or *y is a successor of x*, these being equivalent locutions. If x precedes y and $x \neq y$, then we say that *x immediately precedes y* or x is an *immediate predecessor* of y, etc., just in case there is no element z distinct from both x and y such that x precedes z and z precedes y. In other words, there is no other element between x and y in the order. Note that no element can be said to immediately precede itself since x and y in the definition must be distinct.

In R_1 and S_1 in (3-21) and (3-22), b is between a and c; therefore, although a precedes c, a is not an immediate predecessor of c. In R_2 and S_2, c is an immediate predecessor of b, and b is an immediate predecessor of a.

In diagramming orders it is usually simpler and more perspicuous to connect pairs of elements by arrows only if one is an immediate predecessor of the other. The remaining connection can be inferred from the fact that the relation is transitive. In order to distinguish weak from strict orders, however, it is necessary to include the 'reflexive' loops in weak orders. Diagrammed in this way, the orders in (3-21) would appear as in Figure 3-5. The diagrams of the corresponding strict orders would be identical except for the absence of the loops on each element.

There is also a useful set of terms for elements which stand at the ex-

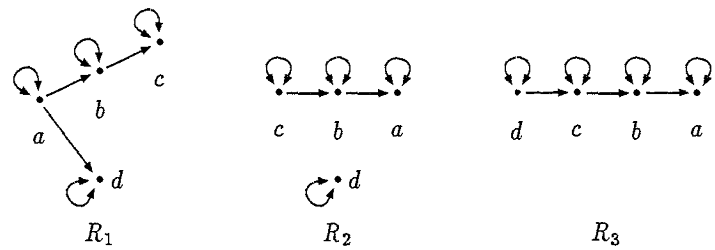

Figure 3–5: Immediate predecessor diagrams
of the orders in (3-21).

tremes of an order. Given an order R in a set A,

1. an element x in A is *minimal* if and only if there is no other element in A which precedes x (examples: a in R_1 and S_1; c and d in R_2 and S_2; d in R_3 and S_3)
2. an element x in A is *least* if and only if x precedes every other element in A (examples: a in R_1 and S_1; d in R_3 and S_3)
3. an element x in A is *maximal* if and only if there is no other element in A which follows x (examples: c and d in R_1 and S_1; a and d in R_2 and S_2; a in R_3 and S_3)
4. an element x in A is *greatest* if and only if x follows every other element in A (examples: a in R_3 and S_3).

Note that a in R_1 and S_1 is both a minimal and a least element, while c and d in these same orders are both maximal but not greatest (c does not follow d, for example). Element d in R_2 and S_2 is both minimal and maximal but neither greatest nor least. The order defined by R in Example (3-13) has 1 as a maximal and greatest element (it follows all other elements and has no successors) but there is no minimal or least element in the order. Observe here that the form 'greatest' as used technically about orders need not coincide with the notions 'greater than ' or 'greatest' in the realm of numbers.

A least element, if there is one in an order, is unique (if there were two, each would have to precede the other, and this would violate either asymmetry or antisymmetry), and similarly for a greatest element. There may be more than one minimal element, however (e.g., c and d in R_2 and S_2 above), and more than one maximal element An order might have none of these; the relation 'greater than' in the set of all positive and negative integers and zero, $\{0, 1, -1, 2, -1, \ldots\}$ has no maximal, minimal, greatest or least elements.

If an order, strict or weak, is also connected, then it is said to be a *total* or *linear* order. Examples are R_3 and S_3 above and the relation R of Example (3-13). Their immediate predecessor diagrams show the elements arranged in a single chain. Order R_1 is not total since d and c are not related, for example. Often orders in general are called *partial orders* or *partially ordered sets*. The terminology is unfortunate, since it then happens that some partial orders are total, but it is well established nonetheless, and we will sometimes use it in the remainder of this book.

Finally, we mention some other frequently encountered notions pertaining to orders A set A is said to be *well-ordered* by a relation R if R is a total order and, further, every subset of A has a least element in the ordering relation. The set of natural numbers $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$ is well-ordered by the 'is less than' relation (it is a total order, and every subset of $\mathbf{N}$ will have a least element when ordered by this relation). The set of integers $\mathbf{Z} = \{0, 1, -1, 2, -2, \ldots\}$, on the other hand, is not well-ordered by that relation, since the negative integers get smaller 'ad infinitum'. Note that every finite linearly ordered set must be well-ordered

A relation R in A is *dense* if for every $\langle x, y \rangle \in R$, $x \neq y$, there exists a member $z \in A$, $x \neq z$ and $y \neq z$, such that $\langle x, z \rangle \in R$ and $\langle z, y \rangle \in R$. Density is an important property of the real numbers which we can think of as all the points lying on a horizontal line of infinite extent. The relation 'is greater than' is not dense on the natural numbers, but it is dense on the real numbers.