Chapter 11

Lattices

11.1 Posets, duality and diagrams

In the previous chapter we have seen that the arithmetical properties of elements of formal systems may be described in operational structures. Operations may serve to generate new elements from a given set of basic elements, and thus we may view an operational or an algebraic structure naturally as a syntactic system which generates elements in a formally precise way. The relation of this dynamic conception of such systems and the linguistic notion of a grammar which generates strings as elements of a natural or formal language will be explored in much more detail in Part E The present chapter is concerned with certain ordering relations between elements of systems or domains of objects and the order-theoretic or 'topological' properties of such ordered structures. We will see that the concepts introduced in this chapter provide a universal perspective on set theory and algebra in which important correlations between the two mathematical theories can be insightfully described. Recently linguistic applications of lattices have been made primarily to semantic topics such as plural NPs, mass terms and events, using the ordering relations to structure the domains of an interpretation of a language. The potential usefulness in linguistics of syntactic applications of lattice theory is explored in research on feature systems, for instance In this chapter we will introduce lattice theory without paying attention to any particular linguistic applications or motivations.

In Chapter 3 we pointed out the set-theoretic importance of partial orderings on sets, i.e. sets of objects ordered by a reflexive, anti-symmetric and transitive relation Here we will call any partially ordered set A together with its ordering $\leq$, i.e. $\langle A, \leq \rangle$, a *poset*, often writing just A and tacitly assuming the intended partial ordering, which is widely accepted practice.

A poset which also satisfies the property of linearity (for all $a, b \in A$: $a \leq b$ or $b \leq a$) is called a *chain*, or a *fully* or *linearly ordered set*. Additional properties and operations may be defined on posets which constitute a stronger structure. Thus the real numbers form a poset, but also a chain, disregarding the arithmetical operations.

If A is a poset and $a, b \in A$, then a and b are *comparable elements* or *comparable objects* if $a \leq b$ or $b \leq a$. If a and b are not comparable, they are *incomparable*, written as $a \parallel b$. In a chain there are of course no incomparable elements.

In an arbitrary poset A we define an upper bound of $B \subseteq A$ as an element $a \in A$, if it exists, such that for all $b \in B$, $b \leq a$. An upper bound a of B is *the least upper bound of* B (abbreviated to *lub* of B) or *the supremum of* B (abbreviated to sup B) if, for any upper bound c of B, we have $a \leq c$. We often write $a = \bigvee B$, or $a =$sup B, since by antisymmetry of the ordering relation we know that if B has a least upper bound, this is a unique least upper bound.

If $\langle A, \leq \rangle$ is a poset, then inversion of the partial ordering preserves the poset characteristics, i.e. writing $a \geq b$ for $b \leq a$ in the given poset we have defined a new poset $\langle A, \geq \rangle$. Verification of the three requirements on a partial order in this new poset $\langle A, \geq \rangle$ is straightforward: e.g., antisymmetry holds since if $a \geq b$ and $b \geq a$, the definition of $\geq$ gives us $b \leq a$ and $a \leq b$, and we know that in $\langle A, \leq \rangle$ in that case $a = b$. We call $\langle A, \geq \rangle$ the *dual* of $\langle A, \leq \rangle$, which is obviously a symmetric relation between posets. This notion will come in handy in proving statements about posets, since it allows us to replace all occurrences of $\leq$ in a true statement S by $\geq$, thus obtaining the (equally true) dual S' of S, without actually carrying out the entire proof for the inverse of the partial ordering.

To appreciate the importance of this duality in posets, we define the dual of an upper bound of $B \subseteq A$, called a *lower bound of* $B \subseteq A$, as an element $a \in A$ such that for all $b \in B$, $b \geq a$ which is equivalent to $a \leq b$. A lower bound a of B is *the greatest lower bound of* B (abbreviated to glb of B) or *the infimum of* B (abbreviated to inf B) if, for any lower bound c of B we have $a \geq c$. We write $a = \bigwedge B$, or $a = \inf B$. Supremum and infimum are thus duals; hence whatever we may prove about one of them holds also of the other. For instance, we proved above that if a subset B in a poset has a supremum, it has a unique supremum, so we know by duality that the infimum of B, if its exists, is unique.

Partial orderings may be represented visually by so called Hasse diagrams The *diagram* of a poset $\langle A, \leq \rangle$ represents the elements or objects by $\circ$, and if the ordering relation holds between two elements, they are connected by a line, reflecting the order from bottom to top in the representation

For instance, writing out the ordering set-theoretically, let the poset $A = \{\langle 0,0\rangle, \langle 0,a\rangle, \langle 0,b\rangle, \langle 0,1\rangle, \langle a,a\rangle, \langle a,1\rangle, \langle b,b\rangle, \langle b,1\rangle, \langle 1,1\rangle\}$. Assuming reflexivity and transitivity of the connecting lines, we represent A by the diagram in Figure 11-1 (cf. the immediate successor diagrams of Section 3-5)

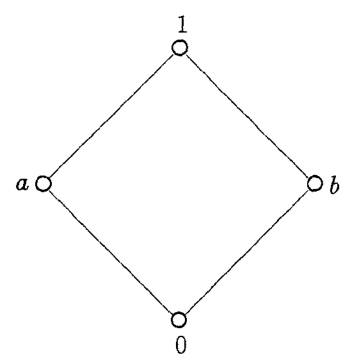


Figure 11–1: The diagram of a poset
$A = \langle\{0, a, b, 1\}, \leq \rangle$.

We call a diagram *planar* or *flat* if it does not contain any intersecting lines, as in Figure 11-1. In general greater clarity of representation is obtained if the number of intersecting lines is kept as small as possible. From Figure 11-1 we can read off that $0 \leq 1$ since we assumed transitivity of the connecting lines, and also we generally know that $x \leq x$ for any arbitrary element x.

We say that *a covers b* (or that *b is covered by a*) if $a > b$ and for no $c : a > c > b$ (Recall that $a < b$ means $a \leq b$ and $a \neq b$.) By induction on the length of chains, we may prove that the covering relation determines the partial ordering in a *finite* poset.

Diagrams usually represent finite posets, but infinite posets are sometime partially represented by diagrams and need further explanation in the text. Note that the real and the rational numbers, despite their essential order-theoretic differences, are represented by the same linear diagram, due to the 'poverty' of the covering relation which determines the diagram. Dualization of a given poset turns the diagram upside down, but preserves the connecting lines.

Set-theoretic inclusion induces a natural partial order on the power set of a given set A, i.e., $\wp(A)$ is a poset. We represent this inclusion relation on the power set for the set $A = \{a, b, c\}$ in Figure 11-2.

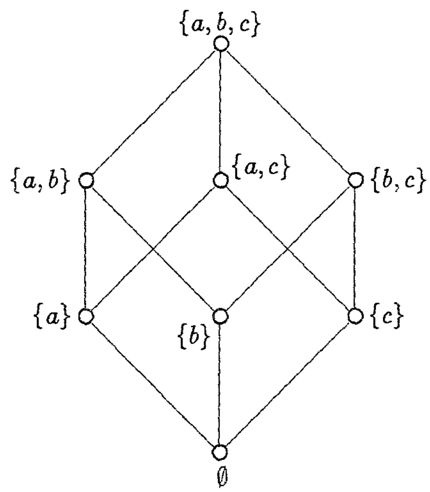


Figure 11-2: The diagram of the poset $\wp(A)$ for $A = \{a, b, c\}$.

Intersecting lines may not define an element. In Figure 11-3 a poset is represented in which all pairs of elements have an upper and lower bound, but these are not always unique. E.g., both c and d are upper bounds for $\{a, b\}$, but neither c nor d is a supremum for $\{a, b\}$.

11.2 Lattices, semilattices and sublattices

There is a special class of posets, called *lattices*, which have proven to be very important in the general study of a variety of mathematical theories including analysis, topology, logic, algebra and geometry. They have led to many

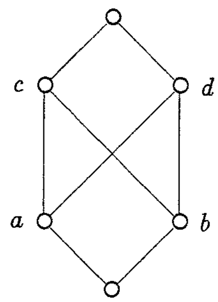


Figure 11–3: The diagram of a poset with non-unique upper and lower bounds

fruitful interactions and new results in various theories and to a productive development of universal algebra and more recently category theory. Linguistic applications of lattice theory are currently being developed in syntax and semantics.

There are two ways of defining lattices, one from a given poset and the other, more in line with the group-theoretic definitions, by requiring properties on operations. We present the two definitions in this order.

A poset $\langle A, \leq \rangle$ is a *lattice* if $\sup\{a, b\}$ and $\inf\{a, b\}$ exist for all $a, b \in A$. We will introduce two new operations $a \wedge b = \inf\{a, b\}$, calling $a \wedge b$ the *meet* of a and b, and $a \vee b = \sup\{a, b\}$, calling $a \vee b$ the *join* of a and b. In lattices the operations of meet and join are always binary, i.e., we may view them dynamically as maps from $A \times A$ to A. This allows us to characterize a lattice as an algebra, i.e. as a non-empty set with two operations with certain algebraic properties. The four properties characteristic of lattice operations are:

(L1) $a \wedge a = a, a \vee a = a$ — idempotent law

(L2) $a \wedge b = b \wedge a, a \vee b = b \vee a$ — commutative law

(L3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ — associative laws
$(a \vee b) \vee c = a \vee (b \vee c)$

The important fourth property of lattice operations connects the two operations. Note first that if $a \leq b$, then $\inf\{a, b\} = a$, i.e. $a \wedge b = a$, and dually, if $a \geq b$, then $\sup\{a,b\} = a$, i e , $a \vee b = a$. Since $a \leq a \vee b$ by definition of $\sup\{a,b\}$, we let $a \vee b$ substitute for b in the first equations to derive $a \wedge (a \vee b) = a$. Similarly, since $a \geq a \wedge b$ by definition of $\inf\{a,b\}$, we derive from the second equations $a \vee (a \wedge b) = a$. Thus we have the two absorption laws:

(L4) $a \wedge (a \vee b) = a$ absorption law
$a \vee (a \wedge b) = a$

Any algebra with two binary operations that have these four properties (L1)–(L4) constitutes a lattice. It will often be very useful to view lattices as algebras, since all that we know about algebraic structures can readily be transferred to lattices. In fact, we often make use of the following theorem, provable from (L1)–(L4), about the connection between lattices represented as posets and lattices represented as algebras.

THEOREM 11.1

(i) Let the poset $\mathbf{A} = \langle A, \leq \rangle$ *be a lattice. Set* $a \wedge b = \inf\{a,b\}$ *and* $a \vee b = \sup\{a,b\}$. *Then the algebra* $\mathbf{A^a} = \langle A, \wedge, \vee \rangle$ *is a lattice.*

(ii) Let the algebra $\mathbf{A} = \langle A, \wedge, \vee \rangle$ *be a lattice. Set* $a \leq b$ *iff* $a \wedge b = a$. *Then* $\mathbf{A^p} = \langle A, \leq \rangle$ *is a poset and the poset* $\mathbf{A^p}$ *is a lattice.*

(iii) Let the poset $\mathbf{A} = \langle A, \leq \rangle$ *be a lattice. Then* $(\mathbf{A^a})^\mathbf{p} = \mathbf{A}$.

(iv) Let the algebra $\mathbf{A} = \langle A, \wedge, \vee \rangle$ *be a lattice. Then* $(\mathbf{A^p})^\mathbf{a} = \mathbf{A}$.

■

Proof: (i) We leave it to the reader to verify that the meet and join operations as defined in (i) satisfy (L1)–(L4). Absorption becomes $a = \sup\{a, \inf\{a,b\}\}$, which is clearly true since $\inf\{a,b\} \leq a$.
(ii) From $a \wedge a = a$ follows $a \leq a$ (reflexive). If $a \leq b$ and $b \leq a$ then $a \wedge b = a$ and $b \wedge a = b$; hence $a = b$ (anti-symmetry). If $a \leq b$ and $b \leq c$ then $a \wedge b = a$ and $b \wedge c = b$, so $a = a \wedge b = a \wedge (b \wedge c) = (a \wedge b) \wedge c = a \wedge c$; hence $a \leq c$ (transitivity). I.e., $\leq$ is a partial order on A. To show that this poset is a lattice we verify existence of sup and inf for any a, b in A. From $a = a \wedge (a \vee b)$ and $b = b \wedge (a \vee b)$ follows $a \leq a \vee b$ and $b \leq a \vee b$. So $a \vee b$ is an upper bound of both a and b. We now want to show that it is a least upper bound, i.e., if for some c, $a \leq c$ and $b \leq c$, then $a \vee b \leq c$. Suppose

$a \leq c$ and $b \leq c$ then $a \vee c = (a \wedge c) \vee c = c$ and similarly for $b \vee c = c$, so $(a \vee c) \vee (b \vee c) = c \vee c = c$. Hence $(a \vee b) \vee c = c$. Absorption gives us $(a \vee b) \wedge c = (a \vee b) \wedge [(a \vee b) \vee c] = a \vee b$, i.e. $a \vee b \leq c$. Thus $a \vee b = \sup\{a, b\}$. Dual reasoning gives us $a \wedge b = \inf\{a, b\}$
(iii) and (iv) check to see that the orderings in $(\mathbf{A}^{\mathbf{a}})^{\mathbf{p}}$, $\mathbf{A}$ and $(\mathbf{A}^{\mathbf{p}})^{\mathbf{a}}$ are the same. ■

These facts guarantee a smooth transition between lattices as posets and as algebras. We may choose whatever perspective is most convenient for our purposes, while knowing that all results will be preserved when the same lattice is represented differently

Duality in lattices as algebras is simply obtained by exchanging the two operations in any statement about lattices.

Next we consider parts of the structure of a lattice and we will see that the algebraic definition and the order-theoretic definition of a lattice show some discrepancy concerning their substructures.

If $\mathbf{L}$ is a lattice and $\mathbf{L}'$ is a non-empty subset of $\mathbf{L}$ such that for every pair of elements a, b in $\mathbf{L}'$ both $a \wedge b$ and $a \vee b$ are in $\mathbf{L}'$ (where $\wedge$ and $\vee$ are the lattice operations of $\mathbf{L}$), then $\mathbf{L}'$ with the same operations restricted to $\mathbf{L}'$ is a *sublattice* of $\mathbf{L}$. If $\mathbf{L}'$ is a sublattice of $\mathbf{L}$, then for any a, b in $\mathbf{L}'$ $a \leq b$ is in $\mathbf{L}'$ iff $a \leq b$ is in $\mathbf{L}$. But note that for a given lattice $\mathbf{L}$ there may be subsets which as posets are lattices, but which do not preserve the meets and joins of $\mathbf{L}$, and hence are not sublattices of $\mathbf{L}$. An example is given in Figure 11-4 where $\mathbf{L} = \langle\{a, b, c, d, e\}, \leq\rangle$ and $\mathbf{L}' = \langle\{a, c, d, e\}, \leq'\rangle$, which is a lattice as poset but which is not a sublattice of $\mathbf{L}$, since in $\mathbf{L}$ $c \vee d = b$ whereas in $\mathbf{L}'$ $c \vee d = a$.

In the next section we will come to understand the reason for defining the sublattice notion algebraically, rather than on the poset representation of a lattice. For the present it suffices to note that the algebraic sublattice notion is stronger than the sub-poset which is also a lattice. It is important to realize that the above theorem about the equivalences between poset representation and algebraic representation of a lattice may break down once we have to consider parts of their structure. There are lattice-theoretic structures which are 'weaker' in the sense of representing just parts of a lattice with less of its structure. The following notions are special cases of sublattices called *semilattices* A poset is a *join semilattice* if $\sup\{a, b\}$ exists for any elements a, b. Dually, a poset is a *meet semilattice* if $\inf\{a, b\}$ exists for any a, b. In a diagram, conventionally, a join semilattice is represented top-down, and a meet semilattice bottom-up. There are again equivalent

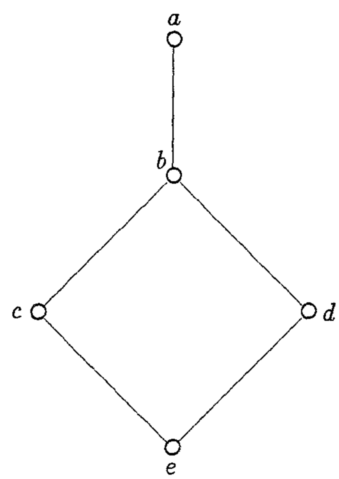


Figure 11–4: The sub-poset $\mathbf{L}' = \langle \{a, c, d, e\}, \leq \rangle$ is a lattice, and a subalgebra, but not a sublattice.

algebraic definitions: If $\langle A, \circ \rangle$ is an algebra with one binary operation $\circ$, it is a semilattice if $\circ$ is idempotent, commutative and associative. Theorem 11-1 for poset and algebraic representations of lattices holds with the appropriate modifications for both kinds of semilattices.

THEOREM 11.2

(i) *Let the poset* $\mathbf{A} = \langle A, \leq \rangle$ *be a join semilattice Set* $a \vee b = sup\{a, b\}$. *Then the algebra* $\mathbf{A^a} = \langle A, \vee \rangle$ *is a semilattice.*

(ii) *Let the algebra* $\mathbf{A} = \langle A, \circ \rangle$ *be a semilattice. Set* $a \leq b$ *iff* $a \circ b = b$. *Then* $\mathbf{A^p} = \langle A, \leq \rangle$ *is a poset and the poset* $\mathbf{A^p}$ *is a join semilattice.*

(iii) *Let the poset* $\mathbf{A} = \langle A, \leq \rangle$ *be a join semilattice Then* $(\mathbf{A^a})^\mathbf{p} = \mathbf{A}$.

(iv) *Let the algebra* $\mathbf{A} = \langle A, \vee \rangle$ *be a semilattice. Then* $(\mathbf{A^p})^\mathbf{a} = \mathbf{A}$.

■

The proof is deferred to the exercises.

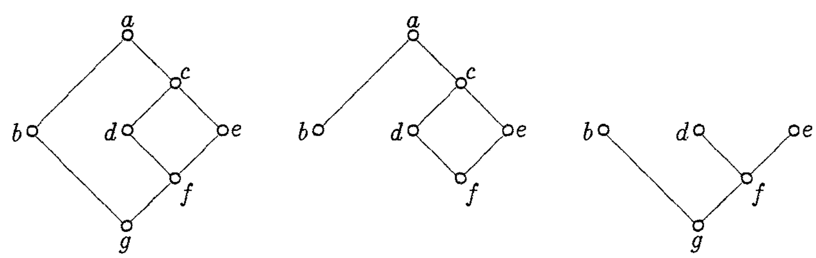


Figure 11–5: A lattice with examples of join and meet semilattices.

11.3 Morphisms in lattices

Mappings from one lattice to another compare their structures, algebraically or order-theoretically.

Two lattices $\mathbf{L_1} = \langle L_1, \wedge, \vee \rangle$ and $\mathbf{L_2} = \langle L_2, \wedge, \vee \rangle$ are *(algebraically) isomorphic* if there is a bijection F from $\mathbf{L_1}$ to $\mathbf{L_2}$ such that for every a, b in L_1

(i) $F(a \vee b) = F(a) \vee F(b)$, and

(ii) $F(a \wedge b) = F(a) \wedge F(b)$

If two lattices are isomorphic, F is called the *lattice isomorphism*. Note that F^{-1} is also a lattice isomorphism, if F is, and that if $F : \mathbf{L_1} \longrightarrow \mathbf{L_2}$ and $F' : \mathbf{L_2} \longrightarrow \mathbf{L_3}$ are lattice isomorphisms, then their composition $F' \circ F : \mathbf{L_1} \longrightarrow \mathbf{L_3}$ is also a lattice isomorphism.

Isomorphism of lattices as posets is defined by requiring the bijection to be order-preserving. If $\mathbf{P_1} = \langle P_1, \leq \rangle$ and $\mathbf{P_2} = \langle P_2, \leq \rangle$ are two posets and $F : \mathbf{P_1} \longrightarrow \mathbf{P_2}$, F is called an *order-preserving map* if $F(a) \leq F(b)$ holds in $\mathbf{P_2}$ whenever $a \leq b$ holds in $\mathbf{P_1}$. Sometimes an order-preserving map is called a *monotone* or an *isotone* mapping.

THEOREM 11.3 *Two posets which are lattices* $\mathbf{L_1} = \langle L_1, \leq \rangle$ *and* $\mathbf{L_2} = \langle L_2, \leq \rangle$ *are* (order-theoretically) isomorphic *iff there is a bijection*

$$F : \mathbf{L_1} \rightarrow \mathbf{L_2} \text{ such that both } F \text{ and } F^{-1} \text{ are order-preserving.}$$

■

Proof. ($\Longrightarrow$) If $F : \mathbf{L_1} \to \mathbf{L_2}$ is an isomorphism and $a \le b$ holds in $\mathbf{L_1}$ then $a = a \wedge b$, so $F(a) = F(a \wedge b) = F(a) \wedge F(b)$, therefore $F(a) \le F(b)$, and F is order-preserving. Dually, the inverse of an order-preserving isomorphism is also order-preserving.
($\Longleftarrow$) Let $F : \mathbf{L_1} \to \mathbf{L_2}$ and its inverse F^{-1} be order-preserving. If a, b in $\mathbf{L_1}$ then $a \le a \vee b$ and $b \le a \vee b$, so $F(a) \le F(a \vee b)$ and $F(b) \le F(a \vee b)$, therefore $F(a) \vee F(b) \le F(a \vee b)$. Suppose $F(a) \vee F(b) \le c$, then $F(a) \le c$ and $F(b) \le c$, and then $a \le F^{-1}(c)$ and $b \le F^{-1}(c)$, so $(a \vee b) \le F^{-1}(c)$ and therefore $F(a \vee b) \le c$. It follows that $F(a) \vee F(b) = F(a \vee b)$ Dually, it is provable that $F(a) \wedge F(b) = F(a \wedge b)$. ■

The diagrams can represent such order-preserving mappings clearly. Figure 11-6 shows an order-preserving bijection $F(a) = a, \ldots, F(d) = d$ from a lattice to a chain which is not an algebraic isomorphism.

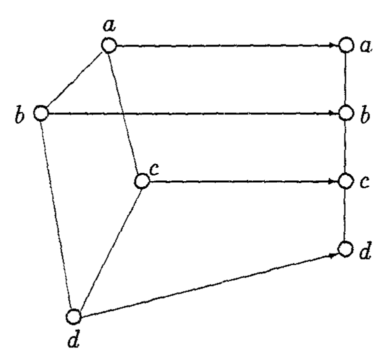


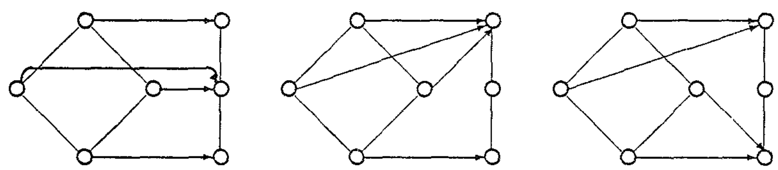
Figure 11–6: An order-preserving bijection
which is not an isomorphism

The following notions are weaker than isomorphisms, and often suffice to characterize the structural similarity between two domains, especially when the mappings are intended to represent information-preserving relations.

A *homomorphism* of the semilattice $\mathbf{S_1} = \langle S_1, \circ \rangle$ into the semilattice $\mathbf{S_2} = \langle S_2, \circ \rangle$ is a mapping $F : \mathbf{S_1} \to \mathbf{S_2}$ such that $F(a \circ b) = F(a) \circ F(b)$. Since any lattice consists of a join and a meet semilattice, this homomorphism notion is split into a *join homomorphism* and a *meet homomorphism*.

A (full) lattice homomorphism is a map that is both a join and a meet homomorphism, i.e. a map F such that $F(a \vee b) = F(a) \vee F(b)$ and $F(a \wedge b) = F(a) \wedge F(b)$.

Note that lattice homomorphisms and join and meet homomorphisms are order-preserving, but the converse is not generally true. In Figure 11-7 the three diagrams show the distinct notions; (11-7 1) is an order-preserving mapping that is neither a join nor a meet homomorphism (cf Figure 11-6), (11-7.2) a join homomorphism that is not a meet homomorphism and (11-7.3) a (full) lattice homomorphism.



(7.1) (7 2) (7.3)

Figure 11-7: An order-preserving mapping, a join homomorphism and a lattice homomorphism

Finally we define an *embedding* of a lattice $\mathbf{L}_1$ into a lattice $\mathbf{L}_2$ as an isomorphism F from $\mathbf{L}_1$ into a sublattice of $\mathbf{L}_2$. If such an embedding exists, $\mathbf{L}_2$ contains a *copy* or an *image* of $\mathbf{L}_1$ as sublattice. This notion will be useful in determining whether a given lattice has a special structure, as we will see below in Section 11 5.

11.4 Filters and ideals

In a lattice we may construct various special subsets with nice properties based on their closure under the ordering relation and the operations.

An *ideal* I of a lattice $\mathbf{L}$ is a non-empty subset of L such that both of the following hold:

(i) if $a \in I$, $b \in L$ and $b \leq a$, then $b \in I$

(ii) if $a, b \in I$, then $(a \vee b) \in I$

An ideal I is *proper* if $I \neq L$ and I is *maximal* if it is not contained in another proper ideal of $\mathbf{L}$.

Dualizing these notions, we define a *filter* F of a lattice $\mathbf{L}$ as a non-empty set of L such that both of the following hold:

(i) if $a \in F$, $b \in L$ and $b \geq a$, then $b \in F$

(ii) if $a, b \in F$, then $(a \wedge b) \in F$

A filter is *proper* if $F \neq L$ and F is maximal if it is not contained in another proper filter of $\mathbf{L}$. Maximal filters are often called *ultrafilters*.

The set of ideals and the set of filters of a lattice are closed under finite intersection, and under arbitrary intersection in case the intersection is not empty (proof is an easy exercise). This *finite intersection property* guarantees existence of the least ideal generated by any non-empty subset $X \subseteq L$, written as $(X]$. If X is a singleton $\{x\} \subseteq L$, then we often write $(x]$ and call this a *principal ideal*. (Dually, the filter $[X)$ generated by $X \subseteq L$, etc.).

If $\mathbf{L}$ is a lattice and $\mathbf{I}(\mathbf{L})$ the set of all ideals in $\mathbf{L}$, then $\mathbf{I}(\mathbf{L})$ is a poset with set inclusion and constitutes a lattice, called the *ideal lattice*. Together with the (provable) claim that any non-empty subset of $\mathbf{I}(\mathbf{L})$ has a supremum, which makes $\mathbf{I}(\mathbf{L})$ a *complete lattice*, we may prove that $\mathbf{L}$ can be embedded in $\mathbf{I}(\mathbf{L})$ by an embedding function $G(x) = (x]$. Sometimes the image of this embedding G is called the *ideal representation* of a lattice (dually, *filter representation*). The proof appeals to a form of the axiom of choice but requires no ingenuity and can be found in any standard reference on lattices (e.g. Grätzer (1971)).

To illustrate this notion of an ideal representation, consider the following simple lattice $\mathbf{L} = \langle\{a, b, c, d\}, \leq\rangle$ in Figure 11-8.
The set of all ideals in $\mathbf{L}$, $\mathbf{I}(\mathbf{L})$, consists of $\{a, b, c, d\}, \{b, c, d\}, \{b, d\}, \{c, d\}$ and the principal ideal $\{d\}$. (Why is e.g. $\{a, b, d\}$ *not* an ideal?) $\mathbf{L}$ can be embedded into $\mathbf{I}(\mathbf{L})$ by the following embedding function: $G : \mathbf{L} \rightarrow \mathbf{I}(\mathbf{L})$ such that
$G(a) = \{a, b, c, d\}$
$G(b) = \{b, d\}$
$G(c) = \{c, d\}$
$G(d) = \{d\}$
The ideal representation of $\mathbf{L}$ is $\{\{a, b, c, d\}, \{b, d\}, \{c, d\}, \{d\}\}$.

The following notions provide 'bounds' to a lattice in a very general way. An element a of a lattice $\mathbf{L}$ is *join-irreducible* if $a = b \vee c$ implies that $a = b$

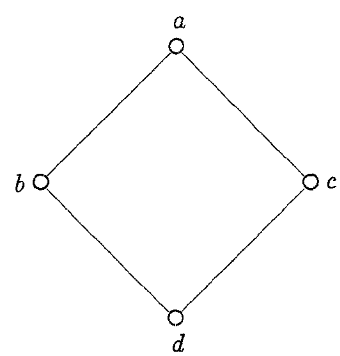


Figure 11–8.

or $a = c$; dually, a is *meet-irreducible* if $a = b \wedge c$ implies that $a = b$ or $a = c$. We call a lattice $\mathbf{L} = \langle L, \wedge, \vee, 0, 1\rangle$ a *bounded lattice* if

(i) $\langle L, \wedge, \vee\rangle$ is a lattice

(ii) $x \wedge 0 = 0$ and $x \vee 1 = 1$, for any arbitrary element x.

These notions will again be useful in Chapter 12

The following is an important theorem establishing a connection between join homomorphisms and ideals.

THEOREM 11.4 *I is a proper ideal of the lattice $\mathbf{L}$ iff there is a join homomorphism G from $\mathbf{L}$ onto the two element chain $\mathbf{C} = \langle\{0,1\}, \leq\ \rangle$ such that $I = G^{-1}(0)$, i.e $I = \{x \mid G(x) = 0\}$.* ■

Proof. $(\Longrightarrow) I$ is a proper ideal and let G be defined by $G(x) = 0$ if $x \in I$ and $G(x) = 1$ if $x \notin I$, which obviously is a join homomorphism.
$(\Longleftarrow)$ If $G : \mathbf{L} \to \mathbf{C}$ is a join homomorphism and $I = G^{-1}(0)$, then for any $a, b \in I$, $G(a) = G(b) = 0$. So $G(a \vee b) = G(a) \vee G(b) = 0 \vee 0 = 0$, hence $(a \vee b) \in I$ If $a \in I$ and $x \in L$ with $x \leq a$, then $G(x) \leq G(a) = 0$, i.e. $G(x) = 0$, so $x \in I$. Furthermore, G is onto, so $I \neq L$, i.e I is proper ■

Of course Theorem 11.4 may be dualized for proper filters.

11.5 Complemented, distributive and modular lattices

In this section we will discuss some particularly well-known lattices which have additional properties and operations providing more structure.

In a bounded lattice **L** we call the least element a, i.e. $a \leq b$ for any $b \in L$, the *bottom* or *zero* of **L**, conventionally writing it as 0. Similarly, the greatest element in a bounded lattice is called the *top* or *unit* of **L**, conventionally written as 1. A bounded lattice $\mathbf{L} = \langle L, \wedge, \vee, 0, 1 \rangle$ is said to be *complemented* if for each $a \in L$ there is a $b \in L$ such that

(C1) $a \vee b = 1$

(C2) $a \wedge b = 0$

and b is called the *complement* of a. In general an element in a lattice may have more than one complement or none at all. A lattice **L** is *relatively complemented* if for any $a \leq b \leq c$ in L there exists d in L with

(RC1) $b \wedge d = a$

(RC2) $b \vee d = c$

and d is called the *relative complement* of b in the interval $[a, c]$. A lattice L is *distributive* if it satisfies either one of the distributive laws

(D1) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(D2) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Since (D1) entails (D2) and vice versa (see exercises), satisfaction of either one suffices for a lattice to be distributive.

It is important to realize that any lattice already satisfies the two lattice inequalities

(LI1) $[(a \wedge b) \vee (a \wedge c)] \leq [a \wedge (b \vee c)]$

(LI2) $[a \vee (b \wedge c)] \leq [(a \vee b) \wedge (a \vee c)]$

Hence to check for distributivity of a lattice it suffices to check the inverses of these inequalities, which together entail (D1) and (D2).

A lattice **L** is *modular* if it satisfies the *modular law*

(M) $a \leq b \rightarrow [a \vee (b \wedge c) = b \wedge (a \vee c)]$

Again since any lattice satisfies $a \leq b \rightarrow [b \wedge (a \vee c) \leq a \vee (b \wedge c)]$ checking the inverse inequality suffices to demonstrate modularity in a lattice.

The following theorem is straightforward.

THEOREM 11.5 *Every distributive lattice is modular* ∎

Proof. If $a \leq b$, $a \vee b = b$ and use this in (D2). ∎

Non-modularity and non-distributivity of a lattice can be verified by embedding two special five element lattices into it, represented by the diagrams NM (Non-Modularity) and ND (Non-Distributivity) in Figure 11-9. These results belong to the core of lattice theory, and are due respectively to Dedekind and to the founder of lattice theory, Birkhoff

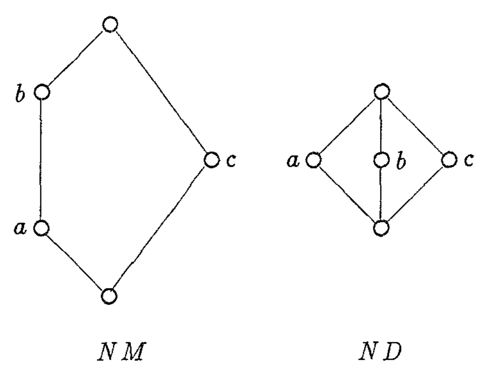
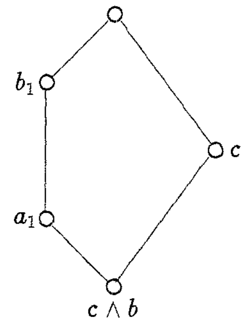


Figure 11–9: Diagrams for non-modularity and non-distributivity

THEOREM 11.6 *(Dedekind)* **L** *is non-modular iff diagram NM can be embedded into* **L** ∎

Proof. ($\Longleftarrow$) In NM $a \leq b$, but it is not the case that $a \vee (b \wedge c) = b \wedge (a \vee c)$, so **L** contains a copy of a non-modular lattice and hence cannot itself be modular.

($\Longrightarrow$) Suppose **L** does not satisfy the modular law, then we will construct a diagram isomorphic to NM as sublattice. For some a, b, c in L we have $a \leq b$, but $a \vee (b \wedge c) < b \wedge (a \vee c)$. Let $a_1 = a \vee (b \wedge c)$ and $b_1 = b \wedge (a \vee c)$.



$$\begin{aligned}
\text{Then } c \wedge b_1 &= c \wedge [b \wedge (a \vee c)] & \\
&= [c \wedge (c \vee a)] \wedge b & \text{commutative, associative laws} \\
&= c \wedge b & \text{absorption} \\
\text{and } c \vee a_1 &= c \vee [a \vee (b \wedge c)] & \\
&= [c \vee (b \wedge c)] \vee a & \text{commutative, associative laws} \\
&= c \vee a & \text{absorption}
\end{aligned}$$

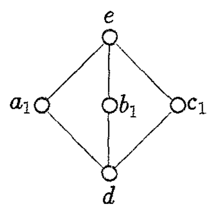
Since $c \wedge b \leq a_1 \leq b_1$ we have $c \wedge b \leq c \wedge a_1 \leq c \wedge b_1 = c \wedge b$, so $c \wedge a_1 = c \wedge b_1 = c \wedge b$ Similarly for $c \vee b_1 = c \vee a_1 = c \vee a$ It is easy to see that the above diagram is isomorphic to and hence a copy of NM ∎

THEOREM 11.7 *(Birkhoff)* **L** *is a non-distributive lattice iff ND can be embedded into* **L** ∎

Proof. ($\Longleftarrow$)$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ does not hold in ND, so if ND can be embedded in **L**, it cannot be a distributive lattice
($\Longrightarrow$) Suppose **L** is non-distributive, i e for some $a, b, c \in L$, $[(a \wedge b) \vee (a \wedge c)] < [a \wedge (b \vee c)]$. Assume also that L does not contain a copy of NM as sublattice, i e , L is modular. Define the following elements, in order to construct a sublattice in L which is isomorphic to ND.

$$\begin{aligned}
d &= (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) \\
e &= (a \vee b) \wedge (a \vee c) \wedge (b \vee c) \\
a_1 &= (a \wedge e) \vee d \\
b_1 &= (b \wedge e) \vee d \\
c_1 &= (c \wedge e) \vee d
\end{aligned}$$

Now $d \leq a_1, b_1, c_1 \leq e$



With absorption we derive from $(a \wedge e) = a \wedge (b \vee c)$, that

$$a \wedge d = a \wedge ((a \wedge b) \vee (a \wedge c) \vee (b \wedge c))$$

Modularity allows exchanging a and $(a \wedge b) \vee (a \wedge c)$

$$\begin{aligned} &= ((a \wedge b) \vee (a \wedge c)) \vee (a \wedge (b \wedge c)) \\ &= (a \wedge b) \vee (a \wedge c) \end{aligned}$$

Now it follows that $d < e$ To show that the above diagram is a copy of ND in L, we prove $a_1 \wedge b_1 = a_1 \wedge c_1 = b_1 \wedge c_1 = d$ and $a_1 \vee b_1 = a_1 \vee c_1 = b_1 \vee c_1 = d$

We prove this here for one case only, the others are similar.

$$\begin{array}{rll} a_1 \wedge b_1 = & ((a \wedge e) \vee d) \wedge ((b \wedge e) \vee d) & \\ & ((a \wedge e) \wedge ((b \wedge e) \vee d) \vee d & \text{modularity} \\ & ((a \wedge e) \wedge ((b \vee d) \wedge e) \vee d & \text{modularity} \\ & ((a \wedge e) \wedge e \wedge (b \vee d)) \vee d & \text{comm. assoc.} \\ & ((a \wedge e) \wedge (b \vee d)) \vee d & \text{idempotent} \\ & (a \wedge (b \vee c) \wedge (b \vee (a \wedge c))) \vee d & \text{absorption} \\ & (a \wedge (b \vee b \vee c) \wedge a \wedge c))) \vee d & \text{modularity} \\ & (a \wedge (b \vee (a \wedge c))) \vee d & a \wedge c \leq b \vee c \\ & (a \wedge c) \vee (b \wedge a) \vee d & \text{modularity} \\ & d & \end{array}$$

■

The following theorems indicate clearly the force of complementation in distributive lattices and correlate it to the weaker notion of relative complementation.

THEOREM 11.8 *In a distributive, complemented lattice each element a has a unique complement* a^*. ■

Proof: Suppose there were two complements a^* and b^* of a, then $a^* = a^* \wedge 1 = a^* \wedge (a \vee b^*) = (a^* \wedge a) \vee (a^* \wedge b^*) = 0 \vee (a^* \wedge b^*) = a^* \wedge b^*$; similarly $b^* = a^* \wedge b^*$, so $a^* = b^*$. ∎

THEOREM 11.9 *In a distributive lattice relative complements are unique, if they exist* ∎

Proof: Let $\mathbf{L}$ be a distributive lattice with $a \le b \le c$ in $\mathbf{L}$ If d and d' were both relative complements of b in the interval $[a, c]$, then

$$\begin{aligned} d = \; & d \wedge c \\ & d \wedge (b \vee d') \\ & (d \wedge b) \vee (d \wedge d') \\ & (d \wedge d') \end{aligned}$$

∎

Similarly, $d' = (d \wedge d')$, so $d = d'$.

THEOREM 11.10 *In a distributive lattice, if a has a complement, then it has a relative complement in any interval containing it.* ∎

Proof: Take $b \le a \le c$ and let d be the complement of a and $x = (d \vee b) \wedge c$ the relative complement of a in $[b, c]$. To prove $a \wedge x = b$ and $a \vee x = c$.

$$a \wedge x = a \wedge ((d \vee b) \wedge c) = ((a \wedge d) \vee (a \wedge b)) \wedge c = (0 \vee b) \wedge c = b,$$

$$a \vee x = a \vee ((d \vee b) \wedge c) = (a \vee d \vee b) \wedge (a \vee c) = 1 \wedge (a \vee c) = c$$

∎

THEOREM 11.11 *In a distributive lattice, if a and b have complements a^* and b^*, respectively, then $a \wedge b$ and $a \vee b$ have complements $(a \wedge b)^*$ and $(a \vee b)^*$, respectively, and the de Morgan identities hold:*

(i) $(a \wedge b)^* = a^* \vee b^*$

(ii) $(a \vee b)^* = a^* \wedge b^*$

∎

Proof: By Theorem 11.8 we only need to prove (i) by verifying

$$(a \wedge b) \wedge (a^* \vee b^*) = 0 \text{ and } (a \wedge b) \vee (a^* \vee b^*) = 1$$

$$(a \wedge b) \wedge (a^* \vee b^*) = (a \wedge b \wedge a^*) \vee (a \wedge b \wedge b^*) = 0 \vee 0 = 0,$$

$$(a \wedge b) \vee (a^* \vee b^*) = (a \vee a^* \vee b^*) \wedge (b \vee a^* \vee b^*) = 1 \wedge 1 = 1$$

The proof of (ii) is an exercise

Exercises

1. Which of the posets in the diagrams of figures 11-1, 11-2 and 11-3 are lattices?

2. (i) Which of the following sets of sentences can be formally represented as posets (each name corresponds to an element):
   (a) Alan is a descendant of Bob and Carol Carol is a descendant of David and Eliza. Bob is a descendant of Fred and Gladys
   (b) as in (a) adding: Fred and Eliza are descendants of Henry and Isabella.
   (c) as in (a) adding: Everyone is a descendant of Adam.
   (d) David and Eliza, who told Fred about it, were told by Bob and Carol, after Alan told them both.
   (e) Jane told Jim and Joseph, who either told Jenny directly or she heard from Julius who had heard from them.

   (ii) Draw diagrams for the posets in (i) and indicate which are semilattices and/or lattices.

   (iii) For the lattices in (i) compute all meets and joins

3. Describe the poset formed by the power set of a four-element set and draw its diagram. What corresponds to the set-theoretic operations in an algebraic representation of this lattice? Check whether they satisfy (L4).

4. Formulate and prove the dual of Theorem 11.2 for meet semilattices.

5. Draw a diagram for a meet homomorphism which is not a join homomorphism from a four-element lattice to a four-element chain and prove it does not represent a lattice homomorphism.

6. Prove that the distributive laws (D1) and (D2) are equivalent.

7. Prove that in a complemented distributive lattice $a = (a^*)^*$.

8. Prove the second of the de Morgan identities of Theorem 11.11.

9. Supply the laws used for each of the proofs of Theorems 11.8–11.11

Chapter 12

Boolean and Heyting Algebras

12.1 Boolean algebras

In this chapter we discuss two well-known algebras as specially structured lattices and prove some of their properties as well as present some semantic interpretations of these structures.

A *Boolean lattice* $\mathbf{BL} = \langle L, \wedge, \vee, *, 0, 1 \rangle$ is a complemented distributive lattice. A *Boolean algebra* is a Boolean lattice in which 0, 1 and $*$ (complementation) are also considered to be operations; i.e. $\mathbf{BA} = \langle B, \wedge, \vee, *, 0, 1\rangle$ where $\vee$ and $\wedge$ are the usual binary operations, $*$ is a unary operation and 0 and 1 are taken to be nullary operations, which simply pick out an element of B. For easy reference, we repeat and relabel the laws which a $\mathbf{BA} = \langle B, \wedge, \vee, *, 0, 1 \rangle$ obeys:

(B0) **BA** is an algebra

(B1) **Associative Laws**

- **(i)** $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
- **(ii)** $a \vee (b \vee c) = (a \vee b) \vee c$

(B2) **Commutative Laws**

- **(i)** $(a \wedge b) = (b \wedge a)$
- **(ii)** $(a \vee b) = (b \vee a)$

(B3) **Distributive Laws**

- **(i)** $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(B4) **Top and Bottom Laws**

(i) $a \wedge 1 = a$ and $a \wedge 0 = 0$

(ii) $a \vee 0 = a$ and $a \vee 1 = 1$

(B5) **Complementation Laws**

(i) $a \wedge a* = 0$

(ii) $a \vee a* = 1$

Often in the literature a special two-element Boolean algebra, called **BOOL** or **2** is used to represent the two truth values 'false' and 'true' where $0 < 1$, $0 = 1*$ and $1 = 0*$

In a **BA** an element a is called an *atom* when a covers 0. The dual notion is less frequently encountered, but defined as an element a which covers 1, called the *dual atom*. A **BA** is called an *atomic* **BA** when it contains an atom a for each of its non-zero elements x such that $a \leq x$. Any finite **BA** is atomic and an atomic **BA** may not be dually atomic.

We prove some central theorems about **BA** which illustrate their power and structural elegance.

THEOREM 12.1 *In a* **BA** *an element is join-irreducible iff it is an atom.* ∎

Proof: ($\Leftarrow$) If a is an atom then $a = b \vee c$ means that $b = a$ or $b = 0$; if $b = 0$ then $a = 0 \vee c = c$; so a is join-irreducible.
($\Rightarrow$) Suppose a is not an atom or 0, then $a > x > 0$ for some element x. When $x < a$, $a = a \wedge 1 = a \wedge (x \vee x*) = (a \wedge x) \vee (a \wedge x*) = x \vee (a \wedge x*)$. Since $a \wedge x* \leq a$ and $a \wedge x* = a$ would imply $x = a \wedge x = a \wedge x * \wedge x = 0$, we know $a \wedge x* < a$, hence a would be join-reducible. ∎

The definitions of ideals and filters in a lattice given in Chapter 11 carry over directly to ideals and filters in **BA**, but note the additional fact that in a **BA** 0 is an element of every ideal and 1 is an element of every filter Due to the strong notion of complementation and the universal top and bottom element in any **BA** we have the following strong correlation between ideals and filters.

THEOREM 12.2 *In any* **BA** *(i) for any $I \subseteq B$, I is an ideal iff $I*$ is a filter; (ii) for any $F \subseteq B$, F is a filter iff $F*$ is an ideal.* ∎

Proof: (i) Note that $0 \in I$ iff $1 = 0* \in I*$ Take $a, b \in I$ then $a \vee b \in I$ iff $(a \vee b)* = a * \wedge b* \in I*$ If $a \in I$, we know $b \leq a$ iff $a* \leq b*$; so $b \in I$ iff $b* \in I*$. The proof of (ii) is obtained dually. ■

THEOREM 12 3 *If F is a filter in a* **BA**, *then F is an ultrafilter iff for each $b \in B$ either $b \in F$ or $b* \in F$* ■

Proof. ($\Leftarrow$) Suppose for any $b \in B$ either $b \in F$ or $b* \in F$ and take F' to be a filter which properly contains F, i e there is some $c \in F' - F$. Since $c \notin F$, $c* \in F \subseteq F'$ So F' is not proper Hence F is a maximal proper filter, an ultrafilter.
($\Rightarrow$) Let F be an ultrafilter and take $b \notin F$. Set $F' = F \cup \{b\}$ which is not proper since F is already maximal So $F \cup \{b\}$ does not have the finite intersection property and for some finite subset X of F, $\inf(X) \wedge b = 0$ So $\inf(X) \leq b*$. $\text{Inf}(X)$ is in F and hence $b* \in F$. ■

The following theorem is proven with a form of the axiom of choice, and shows the existence of a rich class of ultrafilters in any **BA**.

THEOREM 12.4 (The Ultrafilter Theorem) *Each filter in a* **BA** *can be extended to an ultrafilter.* ■

Proof: Let **F** be the non-empty class of all filters in some **BA**, partially ordered by set-theoretic inclusion. We want to show that every chain in this ordering in **F** has an upper bound. Let $C = \{C_i : i \in I\}$ be a chain in **F** and let $\mathbf{C} = \cup_{i \in I} C_i$. If $x, y \in \mathbf{C}$, then for some $i, j \in I$, $x \in C_i$ and $y \in C_j$. Since C is a chain, either $C_i \leq C_j$ or $C_j \leq C_i$; take $C_i \leq C_j$ Then $x, y \in C_j$ and since C_j is a filter $x \wedge y \in C_j \in \mathbf{C}$. If $b \in B$ and $x \leq b$ then $b \in C_j \in \mathbf{C}$. Since $0 \notin C_i$ for any $i \in I$, $0 \notin \mathbf{C}$ So $\mathbf{C}$ is a filter, which is the upper bound for C in **F**. With a form of the axiom of choice (called Zorn's Lemma) we derive that for any filter **F**, **BA** must contain a maximal filter extending that filter. ■

There is an important connection between ultrafilters and homomorphisms, as indicated by the following theorem.

THEOREM 12.5 *Let $\mathbf{BA}_1$ and $\mathbf{BA}_2$ be two Boolean algebras and consider a homomorphism $F : \mathbf{BA}_1 \rightarrow \mathbf{BA}_2$ If U is an ultrafilter of $\mathbf{BA}_2$, then $F^{-1}(U)$ is an ultrafilter of $\mathbf{BA}_1$.* ■

The proof of Theorem 12.5 is not given here, since it requires a number of algebraic concepts which have not been introduced.

12.2 Models of BA

The Boolean laws may already have reminded you of the properties of set-theoretic operations, and, indeed, sets provide simple models of Boolean algebras Starting from any non-empty set X a model for **BA** can be constructed as follows:

- Let B be $\wp(X)$, the power set of X
- Let $\vee$ be set-theoretic union $\cup$
- Let $\wedge$ be set-theoretic intersection $\cap$
- Let $*$ be set-theoretic complementation $'$ relative to X
- Let 1 be X
- Let 0 be $\emptyset$

We may verify that all Boolean laws are true under this interpretation.

(12–1) Let $X = \{a, b, c\}$ then $B = \{\{a, b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{c\}, \emptyset\}$. Union and intersection are as usual and the complements are: $\{a, b, c\}* = \emptyset$, $\{a, b\}* = \{c\}$, $\{a, c\}* = \{b\}$ and $\{b, c\}* = \{a\}$.

Note that by starting from a set with n elements, we construct a **BA** with 2^n elements. Thus for every positive power of 2 there is a Boolean algebra whose set has exactly that cardinality. It can be proven, although we will not do so, that every finite **BA** has a cardinality of 2^n for some positive n. In Section 3 we will prove that every finite model of **BA** is isomorphic to a particular set-theoretic model based on the construction described above. Thus this family of models is particularly important. For infinite models the situation is not so simple. Every infinite set leads to a model for **BA** by the given construction, but not every infinite **BA** is isomorphic to one of these models There are, for instance, Boolean algebras of cardinality $\aleph_0$, but $\aleph_0$ is not the cardinality of the power of any set, as we know from Cantor's Theorem (see Section 4 4).

We can also show that the logic of statements familiar from Part B constitutes a model of **BA**. Let **L** be the logical language whose syntax was defined in Section 2 1, and S be the set of statements generated by its syntactic rules For s and $s' \in S$ we write $s \sim s'$ when s and s' are provably logically equivalent in this logic of statements Now $\sim$ is an equivalence relation on S For each $s \in S$ we define the equivalence class

$$[\![s]\!] = \{s' \in S \mid s \sim s'\}$$

Let B be the set of all such equivalence classes of logically equivalent statements. Define a partial ordering on B by

$$[\![s]\!] \leq [\![s']\!] \text{ iff } (s \rightarrow s') \text{ is valid}$$

Then $\langle B, \leq \rangle$ is a Boolean algebra called the *Lindenbaum algebra* of **L**. The Boolean operations on B are defined by

$$[\![s]\!] \wedge [\![s']\!] = [\![s \,\&\, s']\!], \quad [\![s]\!] \vee [\![s']\!] = [\![s \vee s']\!], \quad [\![s]\!]* = [\![\sim s]\!].$$

Top and bottom are then respectively

$1 = [\![s]\!]$ for any tautology s

$0 = [\![s]\!]$ for any contradiction s

An ultrafilter in the Lindenbaum algebra of **L** can be identified with a maximally consistent set of statements, which would constitute the first step in proving the completeness of **L** through its ultrafilter representation. Such topics belong to more advanced model theory and are beyond the scope of this book (Reference: Bell and Machover (1977)).

12.3 Representation by sets

The first example we gave of a model of **BA** was the power set algebra of a set. In this section we show that each Boolean algebra is isomorphic to a *subalgebra* of a power set algebra, or, in other words, each Boolean algebra may be *reprsented* as a subalgebra of a power set algebra. This important theorem is due to M.F. Stone and is called Stone Representation. We first need to define two new notions:

DEFINITION 12 1

(1) *A* ring *of sets is a family of subsets of a set X which contains for any two subsets A and B of X also $A \cup B$ and $A \cap B$.*

(2) *A* field *of sets is a ring of sets which contains X and the empty set $\emptyset$ and the complement A' of any subset $A \subseteq X$.*

■

From these definitions it is easy to see that a field of subsets of X is a Boolean *subalgebra* of the Boolean power set algebra of X, but that a ring of subsets is a *sublattice* of the power set algebra of X, considered as a distributive lattice. We will prove that any finite distributive lattice is isomorphic with a ring of sets and that any finite Boolean algebra is isomorphic with the field of all subsets of some finite set. (The proof follows essentially Birkhof and MacLane, 377-380).

From Section 12.1 we need Theorem 12.1 and we define a set $S(a) = \{x \mid x \leq a \text{ and } x \text{ covers } 0\}$ of join-irreducible elements x for an element a in any finite lattice **L**. Consider the mapping F which assigns each element a its $S(a)$.

LEMMA 12.1 *In any finite lattice* **L**, *F carries meets in L into set-theoretic intersections: $S(a \wedge b) = S(a) \cap S(b)$.* ■

Proof. By definition of $a \wedge b$ we know that for any join-irreducible element x, $x \leq a \wedge b$ iff $x \leq a$ and $x \leq b$. ■

LEMMA 12.2 *In any finite distributive lattice* **L**, *F carries joins in L into set-theoretic unions: $S(a \vee b) = S(a) \cup S(b)$.* ■

Proof. Take any join-irreducible x, then x is contained in $a \vee b$ iff $x = x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b)$. Now $x \wedge a = x$ or $x \wedge b = x$. So $(a \vee b)$ contains x iff $S(a)$ contains x or $S(b)$ contains x. The converse is obvious in any lattice. ■

These two lemmas show that F is a homomorphism from **L** onto a ring of subsets of the set X of join-irreducible elements of L. Together with the result of Exercise 3 at the end of this chapter we know that F is also a one-to-one onto homomorphism. So we know

THEOREM 12.6 *Any finite distributive lattice is isomorphic with a ring of sets.* ■

In the case of a finite Boolean algebra we know from Theorem 12.1 that each element a is the join of the atoms $x \leq a$. With the above two lemmas we know

$$S(a) \cap S(a') = S(a \wedge a*) = S(0) = \emptyset$$

$$S(a) \cup S(a') = S(a \vee a*) = S(1) = J$$

where J is the set of all join-irreducible element of L. So $[S(a)]* = S(a')$ and F as defined above is an isomorphism from any Boolean algebra to a field of subsets of join-irreducible elements of L. We still need to prove that this field contains *all* sets of join-irreducible elements of L.

THEOREM 12 7 *Any finite Boolean algebra is isomorphic with the Boolean algebra of all sets of its join-irreducible elements* ■

Proof: We need to prove that for any two distinct sets of join-irreducible elements of L the joins of each set are distinct. The claim that the join of all elements in such a set contains all the join-irreducible elements of that set and nothing else follows from

LEMMA 12.3 *If A is a set of join-irreducible elements, and there is some join-irreducible element a such that $a \leq \bigvee\{x \mid x \in A\}$, then $a \in A$.* ■

Proof: $a = a \wedge \bigvee\{x \mid x \in A\} = \bigvee(a \wedge x)$ and since a is join-irreducible for some such $x \in A$, $a \wedge x = a$, so $0 < x \leq a$. But then $a = x$. ■

The significance of Stone Representations for representing information and structuring models for the semantic interpretation of natural language is discussed in Landman (1986). The mathematical import of Boolean algebras can be illustrated further by relating them to certain topological structures and so called Boolean spaces, but the interested reader should consult the exposition of such topics in Grätzer (1971) or Bell and Machover (1977)