8.5.3 Isomorphism

The notion of isomorphism, the relation of "having the same structure", is of fundamental importance in any attempt to set up a concrete model of an abstract system or a mathematical theory of a family of concrete systems. Informally speaking, two systems are isomorphic if some specified part of their structure is identical and they differ only in interpretation or content or in unspecified parts of their structure. For example, a paper pattern for a dress may be said to be isomorphic to the cut-out cloth with respect to size and shape; they differ only in their material. Japanese and Korean are sometimes said to be isomorphic with respect to syntactic structure, a claim which would be true if the two languages differed in their morphemes but sentences could be put into morpheme-by-morpheme correspondence preserving syntactic configurations and permitting the same syntactic operations.

The formal definition applies to a pair of systems A and B, each consisting of a set of elements on which are defined one or more operations and/or one or more relations. (Such systems will be studied in more detail in Part C, where we will look at them as algebras and see many more examples of isomorphisms.)

DEFINITION 8.9 *An isomorphism between two such systems is a one-one correspondence between their elements and a one-one correspondence between their operations and relations which satisfies the following conditions:*

1. *If a relation R holds between two elements of A, the corresponding relation R' holds between the corresponding elements of B; if R does not hold between two elements of A, R' does not hold between the corresponding elements of B*
2. *Whenever corresponding operations are performed on corresponding elements, the results are corresponding elements.*

■

If there exists an isomorphism between two systems A and B, the systems are said to be *isomorphic*. Note that for two systems not to be isomorphic, it must be the case that there is *no* isomorphism between them, not simply that some particular one-one correspondence fails to be an isomorphism.

(8–39) *Examples:*

(1) The set 1,2,3,4,5 with the relation "greater than" ($>$) can be shown to be isomorphic to the set -1,-2,-3,-4,-5 with the relation "less than" ($<$) by letting each number in the first set correspond to its negative in the second set, since for any two positive integers n and n', if $n > n'$, then $-n < -n'$.

(2) The set $A = \{0, 1\}$ with the operation of "absolute value of difference" defined in the first table below is *not* isomorphic to the set $B = \{0, 1\}$ with the operation of ordinary multiplication, shown in the second chart.

A:

	$\lvert x-y\rvert$	y: 0	1
x	0	0	1
	1	1	0

B:

	$x \cdot y$	y: 0	1
x	0	0	0
	1	0	1

Neither of the two possible one-one correspondences between the two sets can give an isomorphism, since (1) in set A, the result of the operation on two of the pairs of elements is one element and on the other two pairs is the other element, while in set B, one element is the result in three cases and the other in only one; and (2) in set B, the result of operating on one element and itself ($0 \cdot 0$ and $1 \cdot 1$) is always that same element, while in set A this is not the case. Either of these reasons alone is actually sufficient to show that no one-one correspondence can be set up so that the operations performed on corresponding elements would yield corresponding elements as results.

Isomorphism plays an important role in model theory. If we ask how many different models there are for a given axiomatic theory, we generally mean different in the sense of non-isomorphic with respect to relevant structure; isomorphic models are alike in relevant structure. In the next section, we will see a model-theoretic application of the notion of isomorphism.

Note that the relation of isomorphism is an equivalence relation in the sense of section 3.4.