8.1.1 Recursive definitions

Consider the set M of all mirror-image strings on $\{a, b\}$. A mirror-image string is one that can be divided into halves, the right half consisting of the same sequence of symbols as the left half but in the reverse order. For example, *aaaa*, *abba*, *babbab*, and *bbabbabb* are mirror-image strings, but *babb*, *aaab*, and *bab* are not. The following is a possible recursive definition of M.

(8–1) 1. $aa \in M \,\&\, bb \in M$
2. $(\forall x)(x \in M \to (axa \in M \,\&\, bxb \in M))$
3. M contains nothing but those members it has by virtue of lines 1 and 2

Line 1, which is called the *base* of the recursive definition, asserts that $x \in M$ is true of the specific string *aa* and *bb*. Line 2, called the *recursion step* or simply the *recursion*, says that for any string x if $x \in M$ is true, then it is also true of the strings formed from x by concatenating an a at both ends or a b at both ends Line 3, the *restriction*, rules out any true instances of $x \in M$ other than those covered by lines 1 and 2. Without the restriction, the definition would specify a class of sets meeting the conditions of lines 1 and 2 but possibly containing other members as well.

The recursion step of a recursive definition is characteristically a conditional in which what is being defined occurs in both the antecedent and the consequent. This makes recursive definitions look like alleged definitions that are circular and, consequently, not really definitions at all. For example, the putative definition of 'subset' in (8-2)

(8–2) For any sets A and B, A is a *subset* of B iff every subset of A is also a subset of B.

contains a vicious circularity in which the notion 'subset' is characterized by appealing to that notion itself. That is, one could not know what a subset is until one had already determined what a subset is. If 'subset' had already been adequately defined in the customary way in terms of the predicate $\in$, then (8-2) would be a perfectly sensible, in fact, true statement; but as a statement introducing the term 'subset' for the first time (8-2) is defectively circular.

In a recursive definition this circularity is avoided by the presence of the base, which makes a nonconditional statement about the thing being defined. Given the base, one can take an appropriate substitution instance of the recursion step and by *Modus Ponens* derive the consequent of that substitution instance. From the base and recursion of (8-1), for example, the following inference can be carried out:

(8–3)	1.	$aa \in M \;\&\; bb \in M$	
	2.	$(\forall x)(x \in M \rightarrow (axa \in M \;\&\; bxb \in M))$	
	3.	$aa \in M$	1, Simp.
	4.	$aa \in M \rightarrow (aaaa \in M \;\&\; baab \in M)$	2, U.I.
	5.	$aaaa \in M \;\&\; baab \in M$	3,4, M.P.
	6.	$baab \in M$	5, Simp.

From this line and another substitution instance of the recursion step

7. $baab \in M \rightarrow (abaaba \in M \;\&\; bbaabb \in M)$ 2, U.I.

we can derive

8. $abaaba \in M \;\&\; bbaabb \in M$ 6, 7, M.P.

Such a series of steps constitutes a proof that certain strings are in M, given the base and recursion of the recursive definition (8-1) as premises. The fact that such a proof is possible for every string asserted to be in M by the definition serves to convince us that this recursive definition really does define something and is not circular. Without the base, however, no such proofs are possible. From the recursion step alone one can derive only a series of conditionals.

(8–4)	1.	$(\forall x)(x \in M \rightarrow (axa \in M \;\&\; bxb \in M))$	
	2.	$aa \in M \rightarrow (aaaa \in M \;\&\; baab \in M)$	1, U.I.
	3.	$(aa \in M \rightarrow aaaa \in M) \;\&\; (aa \in M \rightarrow baab \in M)$	2, Log. Equi
	4.	$aa \in M \rightarrow aaaa \in M$	3, Simp.
	5.	$aaaa \in M \rightarrow (aaaaaa \in M \;\&\; baaaab \in M)$	1, U.I.
	6.	$aa \in M \rightarrow (aaaaaa \in M \;\&\; baaaab \in M)$	4, 5, H.S.

The conclusions that can be derived are statements that *if* certain strings are in M, then so are certain others. Lacking the base, the definition would not assert that M contain any strings at all.

We also note that the close connection between sets and predicates allows us to regard a recursive definition either as defining a predicate, e.g., the predicate 'is a member of M' in the preceding example, or, equivalently, as defining a set that is the extension of that predicate, e.g., the set M.

A slightly more complex example is the recursive definition of the set of well-formed formulas (*wff*'s) in statement logic (cf. Sec. 6.1). The following definition divides those strings constructed from the alphabet

$$C = \{p, q, r, \&, \vee, \sim, \rightarrow, \leftrightarrow, (,)\}$$

that are legitimate expressions in this system of logic, e.g., $((p \,\&\, q) \vee r) \rightarrow s)$, from those, e.g., $(p \,\& \rightarrow r)$, which are not.

(8–5) 1. p is a *wff*; q is a *wff*; r is a *wff*

2. For all α and β, if α and β are *wff*'s then so is

(a) $(\alpha \,\&\, \beta)$

(b) $(\alpha \vee \beta)$

(c) $(\alpha \rightarrow \beta)$

(d) $(\alpha \leftrightarrow \beta)$

(e) $\sim \alpha$

3. Nothing is a *wff* except as a consequence of lines 1 and 2.

Using this definition we can prove that some particular expression, say $((p \,\&\, q) \vee r)$, is a *wff*.

(8–6)	1.	p is a *wff* & q is a *wff*	(1), Simp.
	2.	(p is a *wff* & q is a *wff*) $\rightarrow$ $(p \,\&\, q)$ is a *wff*	(2a), U.I.
	3.	$(p \,\&\, q)$ is a *wff*	1, 2, M.P.
	4.	r is a *wff*	(1), Simp.
	5.	($(p \,\&\, q)$ is a *wff* & r is a *wff*) $\rightarrow$ $((p \,\&\, q) \vee r)$ is a *wff*	(2b), U.I.
	6.	$(p \,\&\, q)$ is a *wff* & r is a *wff*	3, 4, Conj.
	7.	$((p \,\&\, q) \vee r)$ is a *wff*	5, 6, M.P.

The definition in (8-5) does not characterize all the *wff*'s of statement logic since it allows no more than three distinguishable atomic statements p, q, and r. Of course more symbols could be added to the alphabet and the base of the recursive definition could be appropriately expanded, but for any given finite number of symbols for atomic statements there is some *wff* in statement logic containing more than this number of distinct atomic statements. Thus, it would appear that there must be an infinite number of symbols for atomic statements in the alphabet and that the base of the definition must consist of an infinite conjunction of the form p is a *wff* & q is a *wff* & $\cdots$. This raises anew the problem of specifying the members of an infinite set—here, the set of conjuncts in the base of the recursive definition. The solution is to precede the recursive definition of *wff* by a recursive definition of 'atomic statement' (more precisely, the set of symbols denoting atomic statements). One symbol, say p, is chosen and other symbols are created by adding primes successively: p, p', p'', p''', etc. Each such symbol is considered distinct, designating an atomic statement potentially distinct from all others. The recursive definition is as follows:

(8–7) 1. p is (or denotes) an atomic statement
2. For all x, if x is an atomic statement, then so is x'
3. Nothing else is an atomic statement

The recursive definition of *wff* is now as in (8-5) except that the base is replaced by:

1. Every atomic statement is a *wff*.

It is also understood, of course, that the definition of *wff* now applies to strings on the finite alphabet $C' = \{p,', \&, \vee, \sim, \rightarrow, \leftrightarrow, (,)\}$.

Nothing essentially new is involved in framing one recursive definition in terms of another. We have already seen many examples of definitions in which previously defined concepts appear; for example, the definition of 'power set' in terms of 'subset' in Chapter 1. If recursive definition is a legitimate mode of definition, then there can be no objection to using one recursively defined predicate in the recursive definition of another.