## B.2 Special Properties of Relations

Some kinds of relations turn out to be so common that they have been given special names. For instance, $\leq$ and $\subseteq$ both relate their respective domains (say, $\mathbb{N}$ in the case of $\leq$ and $\wp(A)$ in the case of $\subseteq$) in similar ways. To get at exactly how these relations are similar, and how they differ, we categorize them according to some special properties that relations can have. It turns out that (combinations of) some of these special properties are especially important: orders and equivalence relations.

**Definition B.3 (Reflexivity).** A relation $R \subseteq A^2$ is *reflexive* iff, for every $x \in A$, $Rxx$.

**Definition B.4 (Transitivity).** A relation $R \subseteq A^2$ is *transitive* iff, whenever $Rxy$ and $Ryz$, then also $Rxz$.

**Definition B.5 (Symmetry).** A relation $R \subseteq A^2$ is *symmetric* iff, whenever $Rxy$, then also $Ryx$.

**Definition B.6 (Anti-symmetry).** A relation $R \subseteq A^2$ is *anti-symmetric* iff, whenever both $Rxy$ and $Ryx$, then $x = y$ (or, in other words: if $x \neq y$ then either $\neg Rxy$ or $\neg Ryx$).

In a symmetric relation, $Rxy$ and $Ryx$ always hold together, or neither holds. In an anti-symmetric relation, the only way for $Rxy$ and $Ryx$ to hold together is if $x = y$. Note that this does not *require* that $Rxy$ and $Ryx$ holds when $x = y$, only that it isn't ruled out. So an anti-symmetric relation can be reflexive, but it is not the case that every anti-symmetric relation is reflexive. Also note that being anti-symmetric and merely not being symmetric are different conditions. In fact, a relation can be both symmetric and anti-symmetric at the same time (e.g., the identity relation is).

**Definition B.7 (Connectivity).** A relation $R \subseteq A^2$ is *connected* if for all $x, y \in A$, if $x \neq y$, then either $Rxy$ or $Ryx$.

**Definition B.8 (Irreflexivity).** A relation $R \subseteq A^2$ is called *irreflexive* if, for all $x \in A$, not $Rxx$.

**Definition B.9 (Asymmetry).** A relation $R \subseteq A^2$ is called *asymmetric* if for no pair $x, y \in A$ we have both $Rxy$ and $Ryx$.

Note that if $A \neq \emptyset$, then no irreflexive relation on $A$ is reflexive and every asymmetric relation on $A$ is also anti-symmetric. However, there are $R \subseteq A^2$ that are not reflexive and also not irreflexive, and there are anti-symmetric relations that are not asymmetric.

## B.3 Equivalence Relations

The identity relation on a set is reflexive, symmetric, and transitive. Relations $R$ that have all three of these properties are very common.

**Definition B.10 (Equivalence relation).** A relation $R \subseteq A^2$ that is reflexive, symmetric, and transitive is called an *equivalence relation*. Elements $x$ and $y$ of $A$ are said to be *$R$-equivalent* if $Rxy$.

Equivalence relations give rise to the notion of an *equivalence class*. An equivalence relation "chunks up" the domain into different partitions. Within each partition, all the objects are related to one another; and no objects from different partitions relate to one another. Sometimes, it's helpful just to talk about these partitions *directly*. To that end, we introduce a definition:

**Definition B.11.** Let $R \subseteq A^2$ be an equivalence relation. For each $x \in A$, the *equivalence class* of $x$ in $A$ is the set $[x]_R = \{y \in A : Rxy\}$. The *quotient* of $A$ under $R$ is $A/_R = \{[x]_R : x \in A\}$, i.e., the set of these equivalence classes.

The next result vindicates the definition of an equivalence class, in proving that the equivalence classes are indeed the partitions of $A$:

**Proposition B.12.** *If $R \subseteq A^2$ is an equivalence relation, then $Rxy$ iff $[x]_R = [y]_R$.*

*Proof.* For the left-to-right direction, suppose $Rxy$, and let $z \in [x]_R$. By definition, then, $Rxz$. Since $R$ is an equivalence relation, $Ryz$. (Spelling this out: as $Rxy$ and $R$ is symmetric we have $Ryx$, and as $Rxz$ and $R$ is transitive we have $Ryz$.) So $z \in [y]_R$. Generalising, $[x]_R \subseteq [y]_R$. But exactly similarly, $[y]_R \subseteq [x]_R$. So $[x]_R = [y]_R$, by extensionality.

For the right-to-left direction, suppose $[x]_R = [y]_R$. Since $R$ is reflexive, $Ryy$, so $y \in [y]_R$. Thus also $y \in [x]_R$ by the assumption that $[x]_R = [y]_R$. So $Rxy$. □

**Example B.13.** A nice example of equivalence relations comes from modular arithmetic. For any $a$, $b$, and $n \in \mathbb{N}$, say that $a \equiv_n b$ iff dividing $a$ by $n$ gives the same remainder as dividing $b$ by $n$. (Somewhat more symbolically: $a \equiv_n b$ iff, for some $k \in \mathbb{Z}$, $a - b = kn$.) Now, $\equiv_n$ is an equivalence relation, for any $n$. And there are exactly $n$ distinct equivalence classes generated by $\equiv_n$; that is, $\mathbb{N}/_{\equiv_n}$ has $n$ elements. These are: the set of numbers divisible by $n$ without remainder, i.e., $[0]_{\equiv_n}$; the set of numbers divisible by $n$ with remainder 1, i.e., $[1]_{\equiv_n}$; $\dots$; and the set of numbers divisible by $n$ with remainder $n-1$, i.e., $[n-1]_{\equiv_n}$.

## B.4 Orders

Many of our comparisons involve describing some objects as being "less than", "equal to", or "greater than" other objects, in a certain respect. These involve *order* relations. But there are different kinds of order relations. For instance, some require that any two objects be comparable, others don't. Some include identity (like $\leq$) and some exclude it (like $<$). It will help us to have a taxonomy here.

**Definition B.14 (Preorder).** A relation which is both reflexive and transitive is called a *preorder*.

**Definition B.15 (Partial order).** A preorder which is also anti-symmetric is called a *partial order*.

**Definition B.16 (Linear order).** A partial order which is also connected is called a *total order* or *linear order*.

**Example B.17.** Every linear order is also a partial order, and every partial order is also a preorder, but the converses don't hold. The universal relation on $A$ is a preorder, since it is reflexive and transitive. But, if $A$ has more than one element, the universal relation is not anti-symmetric, and so not a partial order.

**Example B.18.** Consider the *no longer than* relation $\preccurlyeq$ on $\mathbb{B}^*$: $x \preccurlyeq y$ iff $\operatorname{len}(x) \leq \operatorname{len}(y)$. This is a preorder (reflexive and transitive), and even connected, but not a partial order, since it is not anti-symmetric. For instance, $01 \preccurlyeq 10$ and $10 \preccurlyeq 01$, but $01 \neq 10$.

**Example B.19.** An important partial order is the relation $\subseteq$ on a set of sets. This is not in general a linear order, since if $a \neq b$ and we consider $\wp(\{a,b\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$, we see that $\{a\} \nsubseteq \{b\}$ and $\{a\} \neq \{b\}$ and $\{b\} \nsubseteq \{a\}$.

**Example B.20.** The relation of *divisibility without remainder* gives us a partial order which isn't a linear order. For integers $n$, $m$, we write $n \mid m$ to mean $n$ (evenly) divides $m$, i.e., iff there is some integer $k$ so that $m = kn$. On $\mathbb{N}$, this is a partial order, but not a linear order: for instance, $2 \nmid 3$ and also $3 \nmid 2$. Considered as a relation on $\mathbb{Z}$, divisibility is only a preorder since it is not anti-symmetric: $1 \mid -1$ and $-1 \mid 1$ but $1 \neq -1$.

**Definition B.21 (Strict order).** A *strict order* is a relation which is irreflexive, asymmetric, and transitive.

**Definition B.22 (Strict linear order).** A strict order which is also connected is called a *strict linear order*.

**Example B.23.** $\leq$ is the linear order corresponding to the strict linear order $<$. $\subseteq$ is the partial order corresponding to the strict order $\subsetneq$.

**Definition B.24 (Total order).** A strict order which is also connected is called a *total order*. This is also sometimes called a *strict linear order*.

Any strict order $R$ on $A$ can be turned into a partial order by adding the diagonal $\mathrm{Id}_A$, i.e., adding all the pairs $\langle x, x \rangle$. (This is called the *reflexive closure* of $R$.) Conversely, starting from a partial order, one can get a strict order by removing $\mathrm{Id}_A$. These next two results make this precise.

**Proposition B.25.** *If $R$ is a strict order on $A$, then $R^+ = R \cup \mathrm{Id}_A$ is a partial order. Moreover, if $R$ is total, then $R^+$ is a linear order.*

*Proof.* Suppose $R$ is a strict order, i.e., $R \subseteq A^2$ and $R$ is irreflexive, asymmetric, and transitive. Let $R^+ = R \cup \mathrm{Id}_A$. We have to show that $R^+$ is reflexive, antisymmetric, and transitive.

$R^+$ is clearly reflexive, since $\langle x, x \rangle \in \mathrm{Id}_A \subseteq R^+$ for all $x \in A$.

To show $R^+$ is antisymmetric, suppose for reductio that $R^+xy$ and $R^+yx$ but $x \neq y$. Since $\langle x, y \rangle \in R \cup \mathrm{Id}_X$, but $\langle x, y \rangle \notin \mathrm{Id}_X$, we must have $\langle x, y \rangle \in R$, i.e., $Rxy$. Similarly, $Ryx$. But this contradicts the assumption that $R$ is asymmetric.

To establish transitivity, suppose that $R^+xy$ and $R^+yz$. If both $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in R$, then $\langle x, z \rangle \in R$ since $R$ is transitive. Otherwise, either $\langle x, y \rangle \in \mathrm{Id}_X$, i.e., $x = y$, or $\langle y, z \rangle \in \mathrm{Id}_X$, i.e.,

$y = z$. In the first case, we have that $R^+yz$ by assumption, $x = y$, hence $R^+xz$. Similarly in the second case. In either case, $R^+xz$, thus, $R^+$ is also transitive.

Concerning the "moreover" clause, suppose $R$ is a total order, i.e., that $R$ is connected. So for all $x \neq y$, either $Rxy$ or $Ryx$, i.e., either $\langle x, y\rangle \in R$ or $\langle y, x\rangle \in R$. Since $R \subseteq R^+$, this remains true of $R^+$, so $R^+$ is connected as well. □

**Proposition B.26.** *If $R$ is a partial order on $X$, then $R^- = R \setminus \mathrm{Id}_X$ is a strict order. Moreover, if $R$ is linear, then $R^-$ is total.*

*Proof.* This is left as an exercise. □

**Example B.27.** $\leq$ is the linear order corresponding to the total order $<$. $\subseteq$ is the partial order corresponding to the strict order $\subsetneq$.

The following simple result which establishes that total orders satisfy an extensionality-like property:

**Proposition B.28.** *If $<$ totally orders $A$, then:*

$$(\forall a, b \in A)((\forall x \in A)(x < a \leftrightarrow x < b) \to a = b)$$

*Proof.* Suppose $(\forall x \in A)(x < a \leftrightarrow x < b)$. If $a < b$, then $a < a$, contradicting the fact that $<$ is irreflexive; so $a \nless b$. Exactly similarly, $b \nless a$. So $a = b$, as $<$ is connected. □