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RELATIONS

1. Relations

Relation. A *relation* from a set S to a set T is a subset of the Cartesian product of S and T. We can put it in symbols like this: R is a relation from S to T iff (R $\subseteq$ (S $\times$ T)). A relation from a set S to a set T is a *binary relation*. Binary relations are the most common relations (at least in ordinary language). Since S $\times$ T is a set of ordered pairs, any relation from S to T is a set of ordered pairs.

For example, let Coins be the set of coins {Penny, Nickel, Dime, Quarter}. Let Values be the set of values {1, . . . 100}. The Cartesian product Coins $\times$ Values is the set of all (coin, value) pairs. One of the subsets of Coins $\times$ Values is the set {(Penny, 1), (Nickel, 5), (Dime, 10), (Quarter, 25)}. This subset of Coins $\times$ Values is a binary relation that associates each coin with its value. It's the is-the-value-of relation. Hence

is-the-value-of $\subseteq$ Coins $\times$ Values.

Domain. If R is a relation from S to T, then the *domain* of R is S. Note that the term *domain* is sometimes used to mean the set of all x in S such that there is some (x, y) in R. We won't use it in this sense.

Codomain. If R is a relation from S to T, then the *codomain* of R is T. The *range* of R is the set of all y in T such that there is some (x, y) in R. The range and codomain are not always the same. Consider the relation is-the-husband-of. The relation associates men with women. The codomain of the relation is the set of women. The range is the set of married women. Since not every woman is married, the range is not the codomain.

Of course, the domain and codomain of a relation may be the same. A relation on a set S is a subset of S $\times$ S. For example, if Human is the set of all humans, then all kinship relations among humans are subsets of the set Human $\times$ Human. As another example, the relation is-a-teacher-of is the set of all (teacher x, student y) pairs such that x is a teacher of y. Of course, we are assuming that x and y are both humans.

There are many notations for relations. If (x, y) is in a relation R, we can write $x\text{R}y$ or $\text{R}(x, y)$ or $x \xrightarrow{R} y$. All these notations are equivalent.

2. Some Features of Relations

Arity. We are not limited to binary relations. We can also define ternary relations. A *ternary relation* is a subset of the Cartesian product S × T × U. A *quaternary relation* is a subset of the Cartesian product S × T × U × W. And so it goes. Generally, an *n-ary relation* is a subset of $S_1 \times S_2 \times \ldots \times S_n$. An *n*-ary relation is also referred to as an *n*-place relation. The *arity* of a relation is the number of its places. So the arity of an *n*-ary relation is *n*. Note that the arity of a relation is sometimes referred to as its *degree*.

Although we are not limited to binary relations, most of the relations we use in philosophical work are binary. Relations of higher arity are scarce. So, unless we say otherwise, the term *relation* just means binary relation.

Inverse. A relation has an inverse (sometimes called a converse). The *inverse* of R is obtained by turning R around. For instance, the inverse of the relation is-older-than is the relation is-younger-than. The inverse of is-taller-than is is-shorter-than. The inverse of a relation R is the set of all (*y*, *x*) such that (*x*, *y*) is in R. We indicate the inverse of R by the symbol R^{-1}. We define the inverse of a relation R in symbols as follows:

$$R^{-1} = \{ (y, x) \mid (x, y) \in R \}.$$

Reflexivity. A relation R on a set S is *reflexive* iff for every *x* in S, (*x*, *x*) is in R. For example, the relation is-the-same-person-as is reflexive. Clark Kent is the same person as Clark Kent. All identity relations are reflexive.

Symmetry. A relation R on S is *symmetric* iff for every *x* and *y* in S, (*x*, *y*) is in R iff (*y*, *x*) is in R. For example, the relation is-married-to is symmetric. For any *x* and *y*, if *x* is married to *y*, then *y* is married to *x*; and if *y* is married to *x*, then *x* is married to *y*. A symmetric relation is its own inverse. If R is symmetric, then $R = R^{-1}$.

Anti-Symmetry. A relation R on S is *anti-symmetric* iff for every *x* and *y* in S, if (*x*, *y*) is in R and (*y*, *x*) is in R, then *x* is identical to *y*. The relation is-a-part-of is anti-symmetric. If Alpha is a part of Beta and Beta is a part of Alpha, then Alpha is identical with Beta. Note that anti-symmetry and symmetry are not opposites. There are relations that are neither symmetric nor anti-symmetric. Consider the relation is-at-least-as-old-as. Since there are many distinct people with the same age, there are cases in which *x* and *y* are distinct; *x* is at least as old as *y*; and *y* is at least as old as *x*. There are cases in which (*x*, *y*) and (*y*, *x*) are in the relation but *x* is not identical to *y*. Thus the relation is not anti-symmetric. But for any *x* and *y*, the fact that *x* is at least as old as *y* does not imply that *y* is at least as old as *x*. Hence the relation is not symmetric.

Transitivity. A relation R on S is *transitive* iff for every *x*, *y*, and *z* in S, if (*x*, *y*) is in R and (*y*, *z*) is in R, then (*x*, *z*) is in R. The relation is-taller-than is transitive. If Abe is taller than Ben, and Ben is taller than Carl, then Abe is taller than Carl.

3. Equivalence Relations and Classes

Partitions. A set can be divided like a pie. It can be divided into subsets that do not share any members in common. For example, the set {Socrates, Plato, Kant, Hegel} can be divided into {{Socrates, Plato}, {Kant, Hegel}}. A division of a set into some subsets that don't share any members in common is a *partition* of that set. Note that {{Socrates, Plato, Kant}, {Kant, Hegel}} is not a partition. The two subsets overlap – Kant is in both. More precisely, a *partition* of a set S is a division of S into a set of non-empty distinct subsets such that every member of S is a member of exactly one subset. If P is a partition of S, then the union of P is S. Thus ∪{{Socrates, Plato}, {Kant, Hegel}} = {Socrates, Plato, Kant, Hegel}.

Equivalence Relations. An *equivalence relation* is a relation that is reflexive, symmetric, and transitive. Philosophers have long been very interested in equivalence relations. Two particularly interesting equivalence relations are *identity* and *indiscernibility.*

If F denotes an attribute of a thing, such as its color, shape, or weight, then any relation of the form is-the-same-F-as is an equivalence relation. Let's consider the relation is-the-same-color-as. Obviously, a thing is the same color as itself. So is-the-same-color-as is reflexive. For any x and y, if x is the same color as y, then y is the same color as x. So is-the-same-color-as is symmetric. For any x, y, and z, if x is the same color as y, and y is the same color as z, then x is the same color as z. So is-the-same-color-as is transitive.

Equivalence Classes. An equivalence relation partitions a set of things into *equivalence classes*. For example, the relation is-the-same-color-as can be used to divide a set of colored things C into sets whose members are all the same color. Suppose the set of colored things C is

$$\{R_1, R_2, Y_1, Y_2, Y_3, G_1, B_1, B_2\}.$$

The objects R_1 and R_2 are entirely red. Each Y_i is entirely yellow. Each G_i is entirely green. Each B_i is entirely blue. The set of all red things in C is $\{R_1, R_2\}$. The things in $\{R_1, R_2\}$ are all color equivalent. Hence $\{R_1, R_2\}$ is one of the color equivalence classes in C. But red is not the only color. Since there are four colors of objects in C, the equivalence relation is-the-same-color-as partitions C into four equivalence classes – one for each color. The partition looks like this:

$$\{\{R_1, R_2\}, \{Y_1, Y_2, Y_3\}, \{G_1\}, \{B_1, B_2\}\}.$$

As a rule, an equivalence class is a set of things that are all equivalent in some way. They are all the same according to some equivalence relation. Given an equivalence relation R, we say the equivalence class of x under R is

$[x]_R = \{ y \mid y$ bears equivalence relation R to $x \}$.

If the relation R is clear from the context, we can just write $[x]$. For example, for each thing x in C, let the color class of x be

$[x] = \{ y \mid y \in C$ & y is the same color as $x \}$.

We have four colors and thus four color classes. For instance,

the red things = $[R_1] = [R_2] = \{R_1, R_2\}$.

We can do the same for the yellow things, the green things, and the blue things. All the things in the color class of x obviously have the same color. So

the partition of C by is-the-same-color-as = $\{ [x] \mid x \in C\}$.

Since no one thing is entirely two colors, no object can be in more than one equivalence class. The equivalence classes are all mutually disjoint. As a rule, for any two equivalence classes A and B, $A \cap B = \{\}$. Since every thing has some color, each thing in C is in one of the equivalence classes. So the union of all the equivalence classes is C. In symbols, $\cup\{ [x] \mid x \in C\} = C$. Generally speaking, the union of all the equivalence classes in any partition of any set A is just A itself.

Equivalence classes are very useful for abstraction. For instance, Frege used equivalence classes of lines to define the notion of an abstract direction (Frege, 1884: 136-39). The idea is this: in ordinary Euclidean geometry, the direction of line A is the same as the direction of line B iff A is parallel to B. The relation is-parallel-to is an equivalence relation. An equivalence class of a line under the is-parallel-to relation is

$[x] = \{ y \mid y$ is a line and y is parallel to $x \}$.

Frege's insight was that we can identify the direction of x with $[x]$. If A is parallel to B, then [A] = [B] and the direction of A is the same as the direction of B. Conversely, if the direction of A is the same as the direction of B, then [A] = [B]; hence A is in [B] and B is in [A]; so A is parallel to B. It follows that the direction of A is the direction of B if, and only if, A is parallel to B.

4. Closures of Relations

We've mentioned three important properties of relations: reflexivity, symmetry, and transitivity. We often want to transform a given relation into a relation that has one or more of these properties. To transform a relation R into a relation with a given property P, we perform the P *closure* of R. For example, to transform a relation R into one that is reflexive, we perform the reflexive closure of R. Roughly speaking, a certain way of closing a relation is a certain way of expanding or extending the relation.

Since equivalence relations are useful, we often want to transform a given relation into an equivalence relation. Equivalence relations are reflexive, symmetric, and transitive. To change a relation into an equivalence relation, we have to make it reflexive, symmetric, and transitive. We have to take its reflexive, symmetric, and transitive closures. We'll define these closures and then give a large example involving personal identity.

Reflexive Closure. We sometimes want to transform a non-reflexive relation into a reflexive relation. We might want to transform the relation is-taller-than into the relation is-taller-than-or-as-tall-as. Since a reflexive relation R on a set X contains all pairs of the form (x, x) for any x in X, we can make a relation R reflexive by adding those pairs. When we make R reflexive, we get a new relation called the *reflexive closure* of R. More precisely,

the reflexive closure of R = $R \cup \{ (x, x) \mid x \in X \}$.

For example, suppose we have the set of people {Carl, Bob, Allan}, and that Carl is taller than Bob and Bob is taller than Allan. We thus have the non-reflexive relation

is-taller-than = { (Carl, Bob), (Bob, Allan) }.

We can change this into the new reflexive relation is-taller-than-or-as-tall-as by adding pairs of the form (x, x) for any x in our set of people. (After all, each person is as tall as himself.) We thereby get the reflexive closure

is-taller-than-or-as-tall-as = { (Carl, Bob), (Bob, Allan),
(Carl, Carl), (Bob, Bob), (Allan, Allan)}.

Symmetric Closure. We sometimes want to transform a non-symmetric relation into a symmetric relation. We can change the relation is-the-husband-of into is-married-to by making it symmetric. We make a relation R symmetric by adding (x, y) to R iff (y, x) is already in R. Of course, when we make R symmetric, we get a new relation – the *symmetric closure* of R. It is defined symbolically like this:

the symmetric closure of R = $R \cup \{ (y, x) \mid (x, y) \in R \}$.

Since { (y, x) | $(x, y) \in R$ } is the inverse of R, which we denoted by R^{-1}, it follows that

the symmetric closure of $R = R \cup R^{-1}$.

For example, suppose we have the set of people {Allan, Betty, Carl, Diane}. Within this set, Allan is the husband of Betty, and Carl is the husband of Diane. We thus have the non-symmetric relation

is-the-husband-of = { (Allan, Betty), (Carl, Diane)}.

We make this into the new symmetric relation is-married-to by taking the pairs in is-the-husband-of and adding pairs of the form (wife y, husband x) for each pair of the form (husband x, wife y) in is-the-husband-of. We thus get the symmetric closure

is-married-to = { (Allan, Betty), (Carl, Diane),
(Betty, Allan), (Diane, Carl)}.

Transitive Closure. We sometimes want to make an intransitive relation into a transitive relation. We do this by taking the *transitive closure* of the relation. The transitive closure is more complex than either the reflexive or symmetric closures. It involves many steps. We'll use the relation is-an-ancestor-of to illustrate the construction of transitive closures.

Since being an ancestor starts with being a parent, we start with parenthood. Indeed, the ancestor relation is the transitive closure of the parenthood relation. For the sake of convenience, we'll let P be the parenthood relation:

P = { (x, y) | x is a parent of y }.

Ancestors include grand parents as well as parents. The grand parent relation is a repetition or iteration of the parent relation: a parent of a parent of y is a grand parent of y. More precisely,

x is a grand parent of y iff
(there is some z)((x is a parent of z) & (z is a parent of y)).

We can put the repetition or iteration of a relation in symbols by using the notion of the *composition* of a relation with itself. It's defined for any relation R like this:

$R \circ R$ = { (x, y) | (there is some z)($(x, z) \in R$ & $(z, y) \in R$ }.

The grand parent relation is obviously the composition of the parent relation with itself. In symbols, is-a-grand-parent-of = $P \circ P$. We can extend this reasoning to great grand parents like this:

x is a great grand parent of y iff
(there is some z)((x is a parent of z) & (z is a grand parent of y)).

The definition of a great grand parent is the composition of the parent relation with itself two times: is-a-great-grand-parent = $P \circ P \circ P$.

When we repeatedly compose a relation with itself, we get the *powers* of the relation:

$R^1 = R$;

$R^2 = R \circ R = R^1 \circ R$;

$R^3 = R \circ R \circ R = R^2 \circ R$;

$R^{n+1} = R^n \circ R$.

In the case of ancestor relations we have

is-a-parent-of	$= P^1$
is-a-grand-parent-of	$= P^2$
is-a-great-grand-parent-of	$= P^3$
is-a-great-great-grand-parent-of	$= P^4$.

And so it goes. We can generalize like this:

is-an-ancestor-n-generations-before $= P^n$.

We've got your ancestors defined by generation. But how do we define your ancestors? We define them by taking the union of all the generations. Your ancestors are your parents unioned with yoŭr grand parents unioned with your great grand parents and so on. Formally

is-an-ancestor-of = $P^1 \cup P^2 \dot{\cup} P^3 \ldots \cup P^n \ldots$ and so on endlessly.

We said the ancestor relation is the transitive closure of the parenthood relation. And we can generalize. Given a relation R, we denote its *transitive closure* by R*. And we define the transitive closure like this:

$R^* = R^1 \cup R^2 \cup R^3 \ldots \cup R^n \ldots$ and so on endlessly.

You might object that the notion of endless unions is vague. And you'd be right. We can make it precise using numbers. Specifically, we use the *natural* *numbers*. These are the familiar counting numbers or whole numbers, starting with 0. And when we say *number*, without any further qualification, we mean natural number. Thus

the transitive closure of R = R* = $\cup\{ R^n \mid n$ is any number $\}$.

An equivalence relation is reflexive, symmetric, and transitive. So we can transform a relation R into an equivalence relation by taking its reflexive, symmetric, and transitive closures. Since we have to take three closures, there are several ways in which we can transform R into an equivalence relation. The order in which we take the symmetric and transitive closures makes a difference.

5. Recursive Definitions and Ancestrals

The transitive closure of a relation is also known as the *ancestral* of the relation. For any relation R, its ancestral is R*. We can define the ancestral of a relation by using a method known as *recursive definition*. A recursive definition involves a friendly circularity. The relation is defined in terms of itself in a logically valid way. Here's how it works with human ancestors:

x is an ancestor of *y* iff
either *x* is a parent of *y*
or there is some *z* such that *x* is a parent of *z* and *z* is an ancestor of *y*.

Observe that is-an-ancestor-of is defined in terms of itself. This sort of loop allows it to be composed with itself endlessly.

Consider the case of grand parents. If *x* is a grand parent of *y*, then there is some *z* such that

x is a parent of *z* and *z* is a parent of *y*.

The fact that *z* is a parent of *y* fits the first clause (the "either" part) of the *ancestor* definition. In other words, every parent is an ancestor. Consequently, we can replace the fact that *z* is a parent of *y* with the fact that *z* is an ancestor of *y* to obtain

x is a parent of some *z* and *z* is an ancestor of *y*.

But this fits the second clause (the "or" part) of the ancestor definition. Hence

x is an ancestor of *y*.

Consider the case of great grand parents. We have

x is a parent of z_1 and z_1 is a parent of z_2 and z_2 is a parent of *y*;

x is a parent of z_1 and z_1 is a parent of z_2 and z_2 is an ancestor of y;

x is a parent of z_1 and z_1 is an ancestor of y;

x is an ancestor of y.

The circularity in a recursive definition allows you to nest this sort of reasoning endlessly. We can do it for great great grand parents, and so on. Here's the general way to give the recursive definition of the ancestral of a relation:

x R* y iff
  either x R y
  or there is some z such that x R z and z R* y.

Ancestrals aren't the only kinds of recursive definitions. Recursive definition is a very useful and very general tool. We'll see many uses of recursion later (see Chapter 8). But we're not going to discuss recursion in general at this time.

7. Closure under an Operation

Closure under an Operation. We've discussed several operations on sets: the union of two sets; the intersection of two sets; the difference of two sets. All these operations are *binary operations* since they take *two* sets as inputs (and produce a third as output). For example, the union operator takes two sets as inputs and produces a third as output. The union of x and y is a third set z. A set S is *closed* under a binary operation $\otimes$ iff for all x and y in S, $x \otimes y$ is also in S.

Any set that is closed under a set-theoretic operation has to be a set of sets. Consider the set of sets S = {{A, B}, {A}, {B}}. This set is closed under the union operator. Specifically, if we take the union of {A, B} with either {A}, {B}, or {A, B}, we get {A, B}, which is in S. If we take the union of {A} with {B}, we get {A, B}, which is in S. So this set is closed under union. But it is not closed under intersection. The intersection of {A} with {B} is the empty set {}. And the empty set is not a member of S.

Given any set X, the power set of X is closed under union, intersection, and difference. For example, let X be {A, B}. Then pow X is {{}, {A}, {B}, {A, B}}. You should convince yourself that pow X is closed under union, intersection, and difference. How do you do this? Make a table whose rows and columns are labeled with the members of pow X. Your table will have 4 rows and 4 columns. It will thus have 16 cells. Fill in each cell with the union of the sets in that row and column. Is the resulting set in pow X? Carry this out for intersection and difference as well.

8. Closure under Physical Relations

An operation is a relation, so we can extend the notion of closure under an operation to closure under a relation. For example, some philosophers say that a universe is a maximal spatio-temporal-causal system of events. This means that the set of events in the universe is closed under all spatial, temporal, and causal relations.

9. Order Relations

Order. A relation R on a set X is an *order relation* iff R is reflexive, anti-symmetric, and transitive. (Note that an order relation is sometimes called a *partial order*.) Since R is reflexive, for all x in X, (x, x) is in R. Since R is anti-symmetric, for all x and y in X, if both (x, y) and (y, x) are in R, then x is identical with y. Since R is transitive, for all x, y, and z in X, if (x, y) is in R and (y, z) is in R, then (x, z) is in R.

An obvious example of an order relation on a set is the relation is-greater-than-or-equal-to on the set of numbers. This relation is symbolized as $\geq$.

Quasi-Order. A relation R on X is a *quasi-order* iff R is reflexive and transitive. (Note that a quasi-order is sometimes called a *pre-order*.) Suppose we just measure age in days – any two people born on the same day of the same year (they have the same birth date) are equally old. Say x is at least as old as y iff x is the same age as y or x is older than y. The relation is-at-least-as-old-as is a quasi-order on the set of persons. It is reflexive. Clearly, any person is at least as old as himself or herself. It is transitive. If x is at least as old as y, and y is at least as old as z, then x is at least as old as z. But it is not anti-symmetric. If x is at least as old as y and y is at least as old as x, it does not follow that x is identical with y. It might be the case that x and y are distinct people with the same birth date. It's worth mentioning that not being anti-symmetric does *not* imply being symmetric. The relation is-at-least-as-old-as is neither anti-symmetric nor symmetric. For if x is younger than y, then y is at least as old as x but x is not at least as old as y.

The difference between order relations and quasi-order relations can be subtle. Consider the relation is-at-least-as-tall-as. Suppose this is a relation on the set of persons, and that there are some distinct persons who are equally tall. The relation is-at-least-as-tall-as is reflexive. Every person is at least as tall as himself or herself. And it is transitive. However, since there are some distinct persons who are equally tall, it is not anti-symmetric. So it is not an order relation. It is merely a quasi-order.

But watch what happens if we restrict is-at-least-as-tall-as to a subset of people who all have distinct heights. In this subset, there are no two people who are equally tall. In this case, is-at-least-as-tall-as remains both reflexive and transitive. Now, since there are no two distinct people x and y who are equally tall, it is always true that if x is at least as tall as y, then y is not at least as tall as x. For if x is at least as tall as y, and there are no equally tall people in the set, then x is taller than y. Consequently, it is always false that ((x is at least as tall as y) & (y is at least as tall as x)). Recall that if the antecedent of a conditional is false, then the conditional is true by default. So the conditional statement (if ((x is at least as tall as y) & (y is at least as tall as x)) then $x = y$) is true by default. So is-at-least-as-tall-as is anti-symmetric on any set of people who all have distinct heights. Therefore, is-at-least-as-tall-as is an order relation on any set of people who all have distinct heights.