Chapter 2

Relations and Functions

2.1 Ordered pairs and Cartesian products

Recall that there is no order imposed on the members of a set. We can, however, use ordinary sets to define an *ordered pair*, written $\langle a, b \rangle$ for example, in which a is considered the *first member* and b is the *second member* of the pair. The definition is as follows:

(2–1) $\langle a, b \rangle =_{def} \{\{a\}, \{a, b\}\}$

The first member of $\langle a, b \rangle$ is taken to be the element which occurs in the singleton $\{a\}$, and the second member is the one which is a member of the other set $\{a, b\}$, but not of $\{a\}$. Now we have the necessary properties of an ordering since in general $\langle a, b \rangle \neq \langle b, a \rangle$. This is so because we have $\{\{a\}, \{a, b\}\} = \{\{b\}, \{a, b\}\}$ (that is, $\langle a, b \rangle = \langle b, a \rangle$), if and only if we have $a = b$. Of course, this definition can be extended to ordered triples and in general ordered n-tuples for any natural number n. Ordered triples are defined as

(2–2) $\langle a, b, c \rangle =_{def} \langle \langle a, b \rangle, c \rangle$

It might have been intuitively simpler to start with ordered sets as an additional primitive, but mathematicians like to keep the number of primitive notions to a minimum.

If we have two sets A and B, we can form ordered pairs from them by taking an element of A as the first member of the pair and an element of B

as the second member. The *Cartesian product* of A and B, written $A \times B$, is the set consisting of all such pairs. The predicate notation defines it as

(2–3) $A \times B =_{def} \{\langle x, y \rangle \mid x \in A \text{ and } y \in B\}$

Note that according to the definition if either A or B is $\emptyset$, then $A \times B = \emptyset$. Here are some examples of Cartesian products:

(2–4) Let $K = \{a, b, c\}$ and $L = \{1, 2\}$, then

$$\begin{aligned} K \times L &= \{\langle a,1\rangle, \langle a,2\rangle, \langle b,1\rangle, \langle b,2\rangle, \langle c,1\rangle, \langle c,2\rangle\} \\ L \times K &= \{\langle 1,a\rangle, \langle 2,a\rangle, \langle 1,b\rangle, \langle 2,b\rangle, \langle 1,c\rangle, \langle 2,c\rangle\} \\ L \times L &= \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\} \end{aligned}$$

It is important to remember that the members of a Cartesian product are *not* ordered with respect to each other. Although each member is an ordered pair, the Cartesian product is itself an unordered set of them.

Given a set M of ordered pairs it is sometimes of interest to determine the smallest Cartesian product of which M is a subset. The smallest A and B such that $M \subseteq A \times B$ can be found by taking $A = \{a \mid \langle a, b \rangle \in M \text{ for some } b\}$ and $B = \{b \mid \langle a, b \rangle \in M \text{ for some } a\}$. These two sets are called the *projections of M onto the first and the second coordinates*, respectively. For example, if $M = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 3,2\rangle\}$, the set $\{1, 3\}$ is the projection onto the first coordinate, and $\{1, 2\}$ the projection onto the second coordinate. Thus $\{1, 3\} \times \{1, 2\}$ is the smallest Cartesian product of which M is a subset.

2.2 Relations

We have a natural understanding of relations as the sort of things that hold or do not hold between objects. The relation 'mother of' holds between any mother and her children but not between the children themselves, for instance. Transitive verbs often denote relations; e.g., the verb 'kiss' can be regarded as denoting an abstract relation between pairs of objects such that the first kisses the second. The subset relation was defined above as a relation between sets. Objects in a set may be related to objects in the same or another set. We write Rab or equivalently aRb if the relation R holds between objects a and b. We also write $R \subseteq A \times B$ for a relation between objects from two sets A and B, which we call a relation *from A to*

B. A relation holding of objects from a single set A is called a relation *in* A. The projection of R onto the first coordinate is called the *domain* of R and the projection of R onto the second coordinate is called the *range* of R. A relation R from A to B thus can be viewed as a subset of the Cartesian product $A \times B$. (There are unfortunately no generally accepted terms for the sets A and B of which the domain and the range are subsets.) It is important to realize that this is a *set-theoretic* reduction of the relation R to a set of ordered pairs, i.e. $\{\langle a, b \rangle \mid aRb\}$. For example, the relation 'mother of' defined on the set H of all human beings would be a set of ordered pairs in $H \times H$ such that in each pair the first member is mother of the second member. We may visually represent a relation R between two sets A and B by arrows in a diagram displaying the members of both sets.

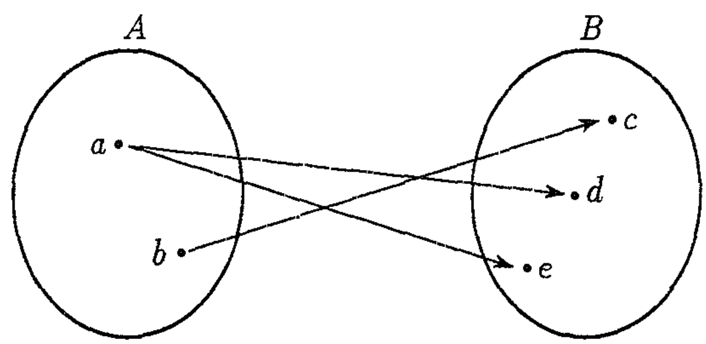

Figure 2–1: Relation R: $A \rightarrow B$.

In Figure 2-1, $A = \{a, b\}$ and $B = \{c, d, e\}$ and the arrows represent a set-theoretic relation $R = \{\langle a, d\rangle, \langle a, e \rangle, \langle b, c\rangle\}$. Note that a relation may relate one object in its domain to more than one object in its range. The complement of a relation $R \subseteq A \times B$, written R', is set-theoretically defined as

(2–5) $R' =_{def} (A \times B) - R$

Thus R' contains all ordered pairs of the Cartesian product which are not members of the relation R. Note that $(R')' = R$. The *inverse* of a relation $R \subseteq A \times B$, written R^{-1}, has as its members all the ordered pairs in R, with their first and second elements reversed. For example, let $A = \{1, 2, 3\}$ and let $R \subseteq A \times A$ be $\{\langle 3, 2 \rangle, \langle 3, 1 \rangle, \langle 2, 1 \rangle\}$, which is the 'greater than' relation in A. The complement relation R' is $\{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 3 \rangle\}$, the 'less than or equal to' relation in A. The inverse of R, R^{-1}, is $\{\langle 2,3\rangle, \langle 1,3\rangle, \langle 1,2\rangle\}$, the 'less than' relation in A. Note that $(R^{-1})^{-1} = R$, and that if $R \subseteq A \times B$, then $R^{-1} \subseteq B \times A$, but $R' \subseteq A \times B$.

We have focused in this discussion on *binary* relations, i.e., sets of ordered pairs, but analogous remarks could be made about relations which are composed of ordered triples, quadruples, etc., i.e., *ternary*, *quaternary*, or just n-place relations.

2.3 Functions

A function is generally represented in set-theoretic terms as a special kind of relation. A relation R from A to B is a function if and only if it meets both of the following conditions:

1. Each element in the domain is paired with just one element in the range.
2. The domain of R is equal to A.

This amounts to saying that a subset of a Cartesian product $A \times B$ can be called a function just in case every member of A occurs exactly once as a first coordinate in the ordered pairs of the set.

As an example, consider the sets $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. The following relations from A to B are functions:

(2–6)
$$\begin{aligned} P &= \{\langle a,1\rangle, \langle b,2\rangle, \langle c,3\rangle\} \\ Q &= \{\langle a,3\rangle, \langle b,4\rangle, \langle c,1\rangle\} \\ R &= \{\langle a,3\rangle, \langle b,2\rangle, \langle c,2\rangle\} \end{aligned}$$

The following relations from A to B are not functions:

(2–7)
$$\begin{aligned} S &= \{\langle a,1\rangle, \langle b,2\rangle\} \\ T &= \{\langle a,2\rangle, \langle b,3\rangle, \langle a,3\rangle, \langle c,1\rangle\} \\ V &= \{\langle a,2\rangle, \langle a,3\rangle, \langle b,4\rangle\} \end{aligned}$$

S fails to meet condition 2 because the set of first members, namely $\{a, b\}$, is not equal to A. T does not satisfy condition 1, since a is paired with both 2 and 3. In relation V both conditions are violated.

Much of the terminology used in talking about functions is the same as that for relations. We say that a function that is a subset of $A \times B$ is a function *from A to B*, while one in $A \times A$ is said to be a function *in A*. The notation '$F: A \to B$' is used for 'F is a function from A to B.' Elements in the domain of a function are sometimes called *arguments* and their correspondents in the range, *values*. Of function P in (2-6), for example, one may say that it takes on the value 3 at argument c. The usual way to denote this fact is $P(c) = 3$, with the name of the function preceding the argument, which is enclosed in parentheses, and the corresponding value to the right of the equal sign.

'Transformation,' 'map,' 'mapping,' and 'correspondence' are commonly used synonyms for 'function,' and often '$F(a) = 2$' is read as 'F maps a into 2.' Such a statement gives a function the appearance of an active process that changes arguments into values. This view of functions is reinforced by the fact that in most of the functions commonly encountered in mathematics the pairing of arguments and values can be specified by a formula containing operations such as addition, multiplication, division, etc. For example, $F(x) = 2x + 1$ is a function which, when defined on the set of integers, pairs 1 with 3, 2 with 5, 3 with 7, and so on. This can be thought of as a rule which says, "To find the value of F at x, multiply x by 2 and add 1." Later in this book it may prove to be necessary to think of functions as dynamic processes transforming objects as their input into other objects as their output, but for the present, we adhere to the more static set-theoretic perspective. Thus, the function $F(x) = 2x + 1$ will be regarded as a set of ordered pairs which could be defined in predicate notation as

(2–8) $F = \{\langle x, y\rangle \mid y = 2x + 1\}$ (where x and y are integers)

Authors who regard functions as processes sometimes refer to the set of ordered pairs obtained by applying the process at each element of the domain as the *graph* of the function. The connection between this use of "graph" and a representation consisting of a line drawn in a coordinate system is not accidental.

We should also note that relations which satisfy condition 1 above but perhaps fail condition 2 are sometimes regarded as functions, but if so, they are customarily designated as 'partial functions.' For example, the function which maps an ordered pair of real numbers $\langle a, b\rangle$ into the quotient of a divided by b (e.g., it maps $\langle 6, 2\rangle$ into 3 and $\langle 5, 2\rangle$ into 2.5) is not defined when $b = 0$. But it is single-valued – each pair for which it is defined is associated with a unique value – and thus it meets condition 1. Strictly speaking, by our definition it is not a function, but it could be called a partial function. A partial function is thus a total function on some subset of the domain. Henceforth, we will use the term 'function,' if required, to indicate a single-valued mapping whose domain may be less than the set A containing the domain.

It is sometimes useful to state specifically whether or not the range of a function from A to B is equal to the set B. Functions from A to B in general are said to be *into* B. If the range of the function equals B, however, then the function is *onto* B. (Thus *onto* functions are also *into*, but not necessarily conversely.) In Figure 2-2 three functions are indicated by the same sort of diagrams we introduced previously for relations generally. It should be apparent that functions F and G are *onto* but H is not. All are of course *into*.

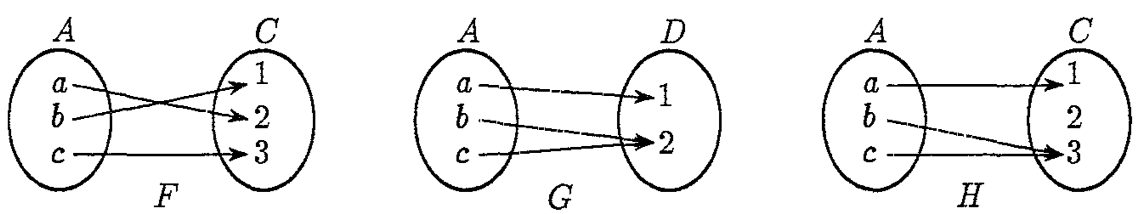

Figure 2–2: Illustration of onto and into functions.

A function $F: A \to B$ is called a *one-to-one* function just in case no member of B is assigned to more than one member of A. Function F in Figure 2-2 is one-to-one, but G is not (since both b and c are mapped into 2), nor is H (since $H(b) = H(c) = 3$). The function F defined in (2-8) is one-to-one since for each odd integer y there is a unique integer x such that $y = 2x + 1$. Note that F is not onto the set of integers since no even integer is the value of F for any argument x. Functions which are not necessarily one-to-one may be termed *many to one*. Thus all functions are many-to-one strictly speaking, and some but not all of them are one-to-one. It is usual to apply the term "many-to-one", however, only to those functions which are not in fact one-to-one.

A function which is both one-to-one and onto (F in Figure 2-2 is an example) is called a *one-to-one correspondence*. Such functions are of special interest because their inverses are also functions (Note that the definitions of the inverse and the complement of a relation apply to functions as well) The inverse of G in Figure 2-2 is not a function since 2 is mapped into both b and c, and in H^{-1} the element 2 has no correspondent.

Problem: Is the inverse of function F in (2-8) also a function? Is F a one-to-one correspondence?

2.4 Composition

Given two functions $F: A \to B$ and $G: B \to C$, we may form a new function from A to C, called the *composite*, or *composition* of F and G, written $G \circ F$. In predicate notation function composition is defined as

(2–9) $G \circ F =_{def} \{\langle x, z \rangle \mid \text{for some } y, \langle x, y \rangle \in F \text{ and } \langle y, z \rangle \in G\}$

Figure 2-3 shows two functions F and G and their composition.

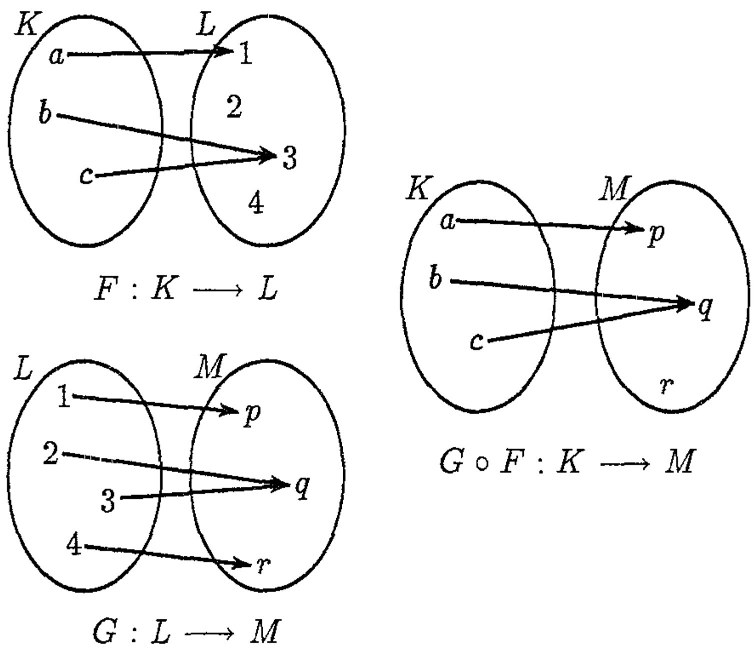

Figure 2–3: Composition of two functions F and G.

Note that F is into while G is onto and that neither is one-to-one. This shows that compositions may be formed from functions that do not have these special properties. It could happen, however, that the range of the first function is disjoint from the domain of the second, in which case, there is no y such that $\langle x, y \rangle \in F$ and $\langle y, z \rangle \in G$, and so the set of ordered pairs defined by $G \circ F$ is empty. In Figure 2-3, F is the first function and G is the second in the composition. Order is crucial here, since in general $G \circ F$ is not equal to $F \circ G$. The notation $G \circ F$ may seem to read backwards, but the value of a function F at an argument a is $F(a)$, and the value of G at the argument $F(a)$ is written $G(F(a))$. By the definition of composition, $G(F(a))$ and $(G \circ F)(a)$ produce the same value.

A function $F: A \rightarrow A$ such that $F = \{\langle x, x \rangle \mid x \in A\}$ is called the *identity function*, written id_A. This function maps each element of A to itself. Composition of a function F with the appropriate identity function gives a function that is equal to the function F itself. This is illustrated in Figure 2-4.

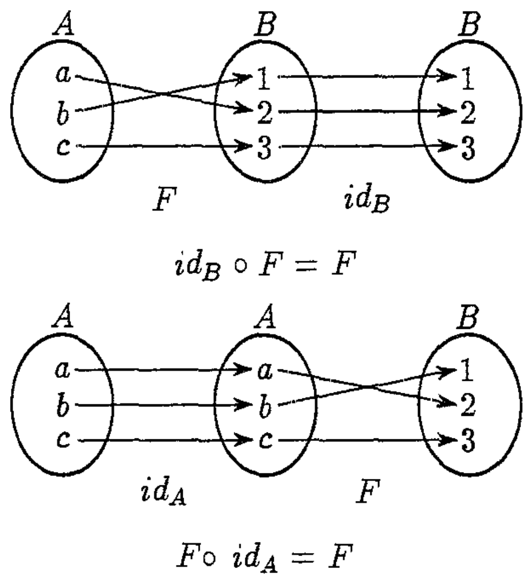

Figure 2–4: Composition with an identity function.

Given a function $F: A \rightarrow B$ that is a one-to-one correspondence (thus the inverse is also a function), we have the following general equations:

$$
\text{(2–10)} \quad
\begin{aligned}
F^{-1} \circ F &= id_A \\
F \circ F^{-1} &= id_B
\end{aligned}
$$

These are illustrated in Figure 2-5.

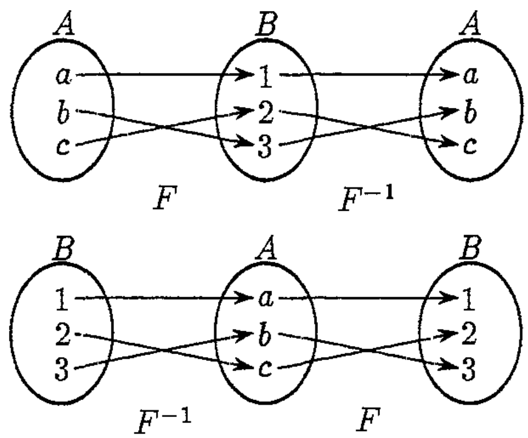

Figure 2–5: Composition of one-to-one correspondence with its inverse.

The definition of composition need not be restricted to functions but can be applied to relations in general. Given relations $R \subseteq A \times B$ and $S \subseteq B \times C$ the composite of R and S, written $S \circ R$, is the relation $\{\langle x, z \rangle \mid$ for some y, $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in S\}$ An example is shown in Figure 2-6.

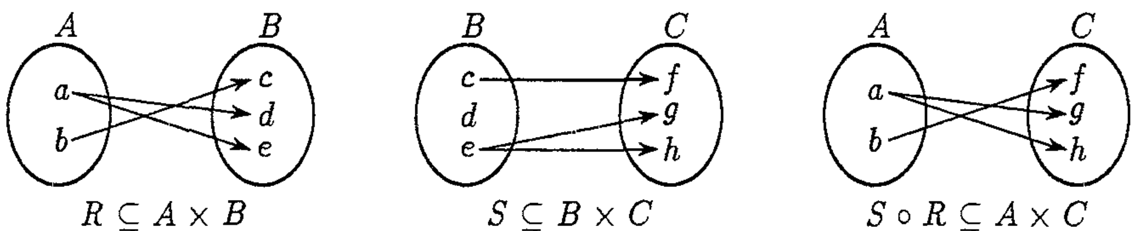

Figure 2–6: Composition of two relations R and S.

For any relation $R \subseteq A \times B$ we also have the following:

$$(2\text{–}11) \quad \begin{aligned} id_B \circ R &= R \\ R \circ id_A &= R \end{aligned}$$

(Note that the identity function in A, id_A, is of course a relation and could equally well be called the identity relation in A)

The equations corresponding to (2-10) do not hold for relations (nor for functions which are not one-to-one correspondences) However, we have for any *one-to-one* relation $R: A \to B$:

(2–12) $$\begin{aligned} R^{-1} \circ R &\subseteq id_A \\ R \circ R^{-1} &\subseteq id_B \end{aligned}$$

We should note here that our previous remarks about ternary, quaternary, etc. relations can also be carried over to functions. A function may have as its domain a set of ordered n-tuples for any n, but each such n-tuple will be mapped into a unique value in the range. For example, there is a function mapping each pair of natural numbers into their sum.

Exercises

1. Let $A = \{b, c\}$ and $B = \{2, 3\}$

(a) Specify the following sets by listing their members.

(i)	$A \times B$	(iv)	$(A \cup B) \times B$
(ii)	$B \times A$	(v)	$(A \cap B) \times B$
(iii)	$A \times A$	(vi)	$(A - B) \times (B - A)$

(b) Classify each statement as true or false.

(i) $(A \times B) \cup (B \times A) = \emptyset$

(ii) $(A \times A) \subseteq (A \times B)$

(iii) $\langle c, c \rangle \subseteq (A \times A)$

(iv) $\{\langle b, 3 \rangle, \langle 3, b \rangle\} \subseteq (A \times B) \cup (B \times A)$

(v) $\emptyset \subseteq A \times A$

(vi) $\{\langle b, 2 \rangle, \langle c, 3 \rangle\}$ is a relation from A to B

(vii) $\{\langle b, b \rangle\}$ is a relation in A

(c) Consider the following relation from A to $(A \cup B)$:
$R = \{\langle b, b \rangle, \langle b, 2 \rangle, \langle c, 2 \rangle, \langle c, 3 \rangle\}$

(i) Specify the domain and range of R

(ii) Specify the complementary relation R' and the inverse R^{-1}

(iii) Is $(R')^{-1}$ (the inverse of the complement) equal to $(R^{-1})'$ (the complement of the inverse)?

2. Let $A = \{a, b, c\}$ and $B = \{1, 2\}$. How many distinct relations are there from A to B? How many of these are functions from A to B? How many of the functions are onto? one-to-one? Do any of the functions have inverses that are functions? Answer the same questions for all relations from B to A.

3. Let

$$R_1 = \{\langle 1,1 \rangle, \langle 2,1 \rangle, \langle 3,4 \rangle, \langle 2,2 \rangle, \langle 3,3 \rangle, \langle 4,4 \rangle, \langle 4,1 \rangle\}$$

$$R_2 = \{\langle 3,4 \rangle, \langle 1,2 \rangle, \langle 1,4 \rangle, \langle 2,3 \rangle, \langle 2,4 \rangle, \langle 1,3 \rangle\}$$

(both relations in A, where $A = \{1, 2, 3, 4\}$).

(a) Form the composites $R_2 \circ R_1$ and $R_1 \circ R_2$. Are they equal?

(b) Show that $R_1^{-1} \circ R_1 \neq id_A$ and that $R_2^{-1} \circ R_2 \not\subseteq id_A$.

4. For the functions F and G in Figure 2-3:

(a) show that $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$.

(b) Show that the corresponding equation holds for relations R and S in Figure 2-6.

Chapter 3

Properties of Relations

3.1 Reflexivity, symmetry, transitivity, and connectedness

Certain properties of binary relations are so frequently encountered that it is useful to have names for them. The properties we shall consider are *reflexivity*, *symmetry*, *transitivity*, and *connectedness*. All these apply only to relations *in* a set, i.e., in $A \times A$ for example, not to relations from A to B, where $B \neq A$.

Reflexivity

Given a set A and a relation R in A, R is *reflexive* if and only if all the ordered pairs of the form $\langle x, x \rangle$ are in R for every x in A.

As an example, take the set $A = \{1, 2, 3\}$ and the relation $R_1 = \{\langle 1, 1 \rangle, \langle 2, 2 \rangle, \langle 3, 3 \rangle, \langle 3, 1 \rangle\}$ in A. R_1 is reflexive because it contains the ordered pairs $\langle 1, 1 \rangle, \langle 2, 2 \rangle$, and $\langle 3, 3 \rangle$. The relation $R_2 = \{\langle 1, 1 \rangle, \langle 2, 2 \rangle\}$ is nonreflexive since it lacks the ordered pair $\langle 3, 3 \rangle$ and thus fails to meet the definitional requirement that it contains the ordered pair $\langle x, x \rangle$ *for every* x *in* A. Another way to state the definition of reflexivity is to say that a relation R in A is reflexive if and only if id_A, the identity relation in A, is a subset of R. The relation 'has the same birthday as' in the set of human beings is reflexive.

A relation which fails to be reflexive is called nonreflexive, but if it contains *no* ordered pair $\langle x, x \rangle$ with identical first and second members, it is said to be *irreflexive*. $R_3 = \{\langle 1, 2 \rangle, \langle 3, 2 \rangle\}$ is an example of an irreflexive relation in A. Irreflexivity is a stronger condition than nonreflexivity since every irreflexive relation is nonreflexive but not conversely. The relation 'is taller than' in the set of human beings is irreflexive (therefore also nonreflexive), while the relation 'is a financial supporter of' is nonreflexive (but not irreflexive, since some people are financially self-supporting) Note that a relation R in A is nonreflexive if and only if $id_A \not\subseteq R$; it is irreflexive if and only if $R \cap id_A = \emptyset$.

Symmetry

Given a set A and a binary relation R in A, R is *symmetric* if and only if for every ordered pair $\langle x, y \rangle$ in R, the pair $\langle y, x \rangle$ is also in R. It is important to note that this definition does not require every ordered pair of $A \times A$ to be in R. Rather for a relation R to be symmetric it must always be the case that *if* an ordered pair is in R, *then* the pair with the members reversed is also in R.

Here are some examples of symmetric relations in $\{1, 2, 3\}$:

(3–1) $\{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 3,2 \rangle, \langle 2,3 \rangle\}$
$\{\langle 1,3 \rangle, \langle 3,1 \rangle\}$
$\{\langle 2,2 \rangle\}$

$\{\langle 2,2 \rangle\}$ is a symmetric relation because for every ordered pair in it, i.e., $\langle 2,2 \rangle$, it is true that the ordered pair with the first and second members reversed, i.e., $\langle 2,2 \rangle$, is in the relation. Another example of a symmetric relation is 'is a cousin of' on the set of human beings. If for some $\langle x, y \rangle$ in R, the pair $\langle y, x \rangle$ is not in R then R is *nonsymmetric*. The relation 'is a sister of' on the set of human beings is nonsymmetric (since the second member may be male. It is, however, a symmetric relation defined on the set of human females).

The following relations in $\{1, 2, 3\}$ are nonsymmetric:

(3–2) $\{\langle 2,3 \rangle, \langle 1,2 \rangle\}$
$\{\langle 3,3 \rangle, \langle 1,3 \rangle\}$
$\{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle, \langle 1,1 \rangle, \langle 2,3 \rangle\}$

If it is *never* the case that for any $\langle x, y \rangle$ in R, the pair $\langle y, x \rangle$ is in R, then the relation is called *asymmetric*. The relation 'is older than' is asymmetric on the set of human beings. Note that an asymmetric relation must be irreflexive (because nothing in the asymmetry definition requires x and y to be distinct). The following are examples of asymmetric relations in $\{1, 2, 3\}$:

(3–3) $\{\langle 2,3\rangle,\langle 1,2\rangle\}$
$\{\langle 1,3\rangle,\langle 2,3\rangle,\langle 1,2\rangle\}$
$\{\langle 3,2\rangle\}$

A relation is *anti-symmetric* if whenever both $\langle x,y\rangle$ and $\langle y,x\rangle$ are in R, then $x = y$. This definition says only that *if* both $\langle x,y\rangle$ and $\langle y,x\rangle$ are in R, then x and y are identical; it does not require $\langle x,x\rangle \in R$ for all $x \in A$. In other words, the relation need not be reflexive in order to be anti-symmetric.

The following relations in $\{1,2,3\}$ are anti-symmetric.

(3–4) $\{\langle 2,3\rangle,\langle 1,1\rangle\}$
$\{\langle 1,1\rangle,\langle 2,2\rangle\}$
$\{\langle 1,2\rangle,\langle 2,3\rangle\}$

Transitivity

A relation R is *transitive* if and only if for all ordered pairs $\langle x,y\rangle$ and $\langle y,z\rangle$ in R, the pair $\langle x,z\rangle$ is also in R.

Because there is no necessity for x, y, and z all to be distinct, the following relation meets the definition of transitivity,

(3–5) $\{\langle 2,2\rangle\}$

where $x = y = z = 2$.

The relation given in (3-6) is not transitive,

(3–6) $\{\langle 2,3\rangle,\langle 3,2\rangle,\langle 2,2\rangle\}$

because $\langle 3,2\rangle$ and $\langle 2,3\rangle$ are members, but $\langle 3,3\rangle$ is not.

Here are some more examples of transitive relations:

(3–7) $\{\langle 1,2\rangle,\langle 2,3\rangle,\langle 1,3\rangle\}$
$\{\langle 1,2\rangle,\langle 2,1\rangle,\langle 1,1\rangle,\langle 2,2\rangle\}$
$\{\langle 1,2\rangle,\langle 2,3\rangle,\langle 1,3\rangle,\langle 3,2\rangle,\langle 2,1\rangle,\langle 3,1\rangle,\langle 1,1\rangle,\langle 2,2\rangle,\langle 3,3\rangle\}$

The relation 'is an ancestor of' is transitive in the set of human beings. If a relation fails to meet the definition of transitivity, it is *nontransitive*. If for no pairs $\langle x,y\rangle$ and $\langle y,z\rangle$ in R, the ordered pair $\langle x,z\rangle$ is in R, then the relation is *intransitive*. For example, the relation 'is the mother of' in the set of human beings is intransitive.

Relation (3-6) is nontransitive, as are the following two:

(3–8) $\{\langle 1,2\rangle, \langle 2,3\rangle\}$
$\{\langle 1,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle, \langle 3,1\rangle\}$

The first of these relations is also intransitive, as are the following relations:

(3–9) $\{\langle 3,1\rangle, \langle 1,2\rangle, \langle 2,3\rangle\}$
$\{\langle 3,2\rangle, \langle 1,3\rangle\}$

Connectedness

A relation R in A is *connected* (or *connex*) if and only if for every two *distinct* elements x and y in A, $\langle x,y\rangle \in R$ or $\langle y,x\rangle \in R$ (or both).

Note that the definition of connectedness refers, as does the definition of reflexivity, to all the members of the set A. Further, the pairs $\langle x,y\rangle$ and $\langle y,x\rangle$ mentioned in the definition are explicitly specified as containing nonidentical first and second members. Pairs of the form $\langle x,x\rangle$ are not prohibited in a connected relation, but they are irrelevant in determining connectedness.

The following relations in $\{1,2,3\}$ are connected:

(3–10) $\{\langle 1,2\rangle, \langle 3,1\rangle, \langle 3,2\rangle\}$
$\{\langle 1,1\rangle, \langle 2,3\rangle, \langle 1,2\rangle, \langle 3,1\rangle, \langle 2,2\rangle\}$

The following relations in $\{1,2,3\}$, which fail the definition, are nonconnected.

(3–11) $\{\langle 1,2\rangle, \langle 2,3\rangle\}$
$\{\langle 1,3\rangle, \langle 3,1\rangle, \langle 2,2\rangle, \langle 3,2\rangle\}$

It may be useful at this point to give some examples of relations specified by predicates and to consider their properties of reflexivity, symmetry, transitivity, and connectedness

(3–12) *Example*: R_f is the relation 'is father of' in the set H of all human beings. R_f is irreflexive (no one is his own father); asymmetric (if x is y's father, then it is never true that y is x's father); intransitive (if x is y's father and y is z's father, then x is z's grandfather but not z's father); and nonconnected (there are distinct individuals x and y in H such that neither 'x is the father of y' nor 'y is the father of x' is true).

(3–13) *Example*: R is the relation 'greater than' defined in the set $Z = \{1, 2, 3, 4, \ldots\}$ of all the positive integers Z contains an infinite number of members and so does R, but we are able to determine the relevant properties of R from our knowledge of the properties of numbers in general. R is irreflexive (no number is greater than itself); asymmetric (if $x > y$, then $y \not> x$; transitive (if $x > y$ and $y > z$, then $x > z$), and connected (for every distinct pair of integers x and y, either $x > y$ or $y > x$.

(3–14) *Example*: R_a is the relation defined by 'x is the same age as y,' in the set H of all living human beings. R_a is reflexive (everyone is the same age as himself or herself); symmetric (if x is the same age as y, then y is the same age as x); transitive (if x and y are the same age and so are y and z, then x is the same age as z); and nonconnected (there are distinct individuals in H who are not of the same age).

3.2 Diagrams of relations

It may be helpful in assimilating the notions of reflexivity, symmetry and transitivity to represent them in relational diagrams. The members of the relevant set are represented by labeled points (the particular spatial arrangement of them is irrelevant). If x is related to y, i.e. $\langle x, y \rangle \in R$, an arrow connects the corresponding points. For example,

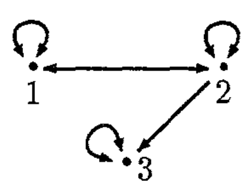

Figure 3–1: Relational diagram.

Figure 3-1 represents the relation

$$R = \{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle, \langle 1,1 \rangle, \langle 2,3 \rangle, \langle 3,3 \rangle\}$$

It is apparent from the diagram that the relation is reflexive, since every point bears a loop. The relation is nonsymmetric since 3 is not related to 2 whereas 2 is related to 3. It cannot be called asymmetric or antisymmetric, however, since 1 is related to 2 and 2 is related to 1. It is nontransitive since 1 is related to 2 and 2 is related to 3, but there is no direct arrow from 1 to 3. The relation cannot be intransitive because of the presence of pairs such as $\langle 1, 1 \rangle$.

If a relation is connected, every pair of distinct points in its diagram will be directly joined by an arrow. We see that R is no connected since there is not direct connection between 1 and 3 in Figure 3-1.

3.3 Properties of inverses and complements

Given that a relation R has certain properties of reflexivity, symmetry, transitivity or connectedness, one can often make general statements about the question whether these properties are preserved when the inverse R^{-1} or complement R' of that relation is formed.

For example, take a reflexive relation R in A. By the definition of reflexive relations, for every $x \in A$, $\langle x, x \rangle \in R$. Since R^{-1} has all the ordered pairs of R, but with the first and second members reversed, then every pair $\langle x, x \rangle$ is also in R^{-1}. So the inverse of R is reflexive also. The complement R' contains all ordered pairs in $A \times A$ that are not in R. Since R contains every pair of the form $\langle x, x \rangle$ for any $x \in A$, R' contains none of them. The complement relation is therefore irreflexive.

As another example, take a symmetric relation R in A. Does its complement have this property? Let's assume that the complement R' is not symmetric, and see what we can derive from that assumption If R' is not symmetric, then there is some $\langle x, y \rangle \in R'$ such that $\langle y, x \rangle \notin R'$, by the definition of a nonsymmetric relation. Since $\langle y, x \rangle \notin R'$, $\langle y, x \rangle$ must be in the complement of R', which is R itself. Because R is symmetric, $\langle x, y \rangle$ must also be in R. But one and the same ordered pair $\langle x, y \rangle$ cannot be both in R *and* in its complement R', so the assumption that the complement R' is not symmetric leads to an absurd conclusion. That means that the assumption cannot be true and the complement R' must be symmetric after all. If R is a symmetric relation in A, then the complement R' is symmetric and vice versa (the latter follows from essentially the same reasoning with R' substituted for R). This mode of reasoning is an instance of what is called a *reductio ad absurdum* proof in logic. It is characterized by making an assumption which leads to a necessarily false conclusion; you may then conclude that the negation of that assumption is true. In Chapter 6 we will introduce rules of inference which will allow such arguments to be made completely precise.

For sake of easy reference the table in Figure 3-2 presents a summary of properties of relations and those of their inverses and complements. These can all be proved on the basis of the definitions of the concepts and the laws of set theory. Since we have not yet introduced a formal notion of proof, we will not offer proofs here, but it is a good exercise to convince yourself of the facts by trying out a few examples, reasoning informally along the lines illustrated above.

R (not $\emptyset$)	R^{-1}	R'
reflexive	reflexive	irreflexive
irreflexive	irreflexive	reflexive
symmetric	symmetric ($R^{-1} = R$)	symmetric
asymmetric	asymmetric	non-symmetric
antisymmetric	antisymmetric	depends on R
transitive	transitive	depends on R
intransitive	intransitive	depends on R
connected	connected	depends on R

Figure 3–2: Preservation of properties of a relation in its inverse and its complement.

3.4 Equivalence relations and partitions

An especially important class of relations are the *equivalence relations.* They are relations which are reflexive, symmetric and transitive. Equality is the most familiar example of an equivalence relation. Other examples are 'has the same hair color as', and 'is the same age as'. The use of equivalence relations on a domain serves primarily to structure a domain into subsets whose members are regarded as equivalent with respect to that relation.

For every equivalence relation there is a natural way to divide the set on which it is defined into mutually exclusive (disjoint) subsets which are called *equivalence classes*. We write $[\![x]\!]$ for the set of all y such that $\langle x, y \rangle \in R$.

Thus, when R is an equivalence relation, $[\![x]\!]$ is the equivalence class which contains x The relation 'is the same age as' divides the set of people into age groups, i.e., sets of people of the same age Every pair of distinct equivalence classes is disjoint, because each person, having only one age, belongs to exactly one equivalence class. This is so even when somebody is 120 years old, and is the only person of that age, consequently occupying an equivalence class all by himself. By dividing a set into mutually exclusive and collectively exhaustive nonempty subsets we effect what is called a *partitioning* of that set.

Given a non-empty set A, a *partition* of A is a collection of non-empty subsets of A such that (1) for any two distinct subsets X and Y, $X \cap Y = \emptyset$ and (2) the union of all the subsets in the collection equals A The notion of a partition is not defined for an empty set. The subsets that are members of a partition are called *cells* of that partition.

For example, let $A = \{a, b, c, d, e\}$. Then, $P = \{\{a, c\}, \{b, e\}, \{d\}\}$ is a partition of A because every pair of cells is disjoint: $\{a, c\} \cap \{b, e\} = \emptyset$, $\{b, e\} \cap \{d\} = \emptyset$, and $\{a, c\} \cap \{d\} = \emptyset$; and the union of all the cells equals A: $\bigcup\{\{a, c\}, \{b, e\}, \{d\}\} = A$.

The following three sets are also partitions of A:

(3–15) $P_1 = \{\{a, c, d\}, \{b, e\}\}$
$P_2 = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}\}$
$P_3 = \{\{a, b, c, d, e\}\}$

P_3 is the trivial partition of A into only one set. Note however that the definition of a partition is satisfied.

The following two sets are not partitions of A:

(3–16) $C = \{\{a, b, c\}, \{b, d\}, \{e\}\}$
$D = \{\{a\}, \{b, e\}, \{c\}\}$

C fails the definition because $\{a, b, c\} \cap \{b, d\} \neq \emptyset$ and D because $\bigcup\{\{a\}, \{b, e\}, \{c\}\} \neq A$

There is a close correspondence between partitions and equivalence relations. Given a partition of set A, the relation $R = \{\langle x, y \rangle \mid x$ and y are in the same cell of the partition$\}$ is an equivalence relation. Conversely, given a reflexive, symmetric, and transitive relation R in A, there exists a partition of A in which x and y are in the same cell if and only if x and y are related by

R. The equivalence classes specified by R are just the cells of the partition. An equivalence relation in A is sometimes said to *induce a partition of A*.

As an example, consider the set $A = \{1, 2, 3, 4, 5\}$ and the equivalence relation

(3–17) $$R = \{\langle 1,1\rangle, \langle 1,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle, \langle 2,2\rangle, \langle 2,4\rangle, \langle 4,2\rangle, \langle 4,5\rangle, \langle 4,4\rangle, \langle 5,2\rangle, \langle 5,4\rangle, \langle 5,5\rangle, \langle 2,5\rangle\}$$

which the reader can verify to be reflexive, symmetric, and transitive. In this relation 1 and 3 are related among themselves in all possible ways, as are 2, 4, and 5, but no members of the first group are related to any member of the second group. Therefore, R defines the equivalence classes $\{1, 3\}$ and $\{2, 4, 5\}$, and the corresponding partition induced on A is

(3–18) $$P_R = \{\{1, 3\}, \{2, 4, 5\}\}$$

Given a partition such as

(3–19) $$Q = \{\{1, 2\}, \{3, 5\}, \{4\}\}$$

the relation R_Q consisting of all ordered pairs $\langle x, y\rangle$ such that x and y are in the same cell of the partition is as follows:

(3–20) $$R_Q = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 3,5\rangle, \langle 5,3\rangle, \langle 5,5\rangle, \langle 4,4$$

R_Q is seen to be reflexive, symmetric, and transitive, and it is thus an equivalence relation.

Another example is the equivalence relation 'is on the same continent as' on the set $A = \{$France, Chile, Nigeria, Ecuador, Luxembourg, Zambia, Ghana, San Marino, Uruguay, Kenya, Hungary$\}$. It partitions A into three equivalence classes: (1) $A_1 = \{$France, Luxembourg, San Marino, Hungary$\}$, (2) $A_2 = \{$Chile, Ecuador, Uruguay$\}$ and (3) $A_3 = \{$Nigeria, Zambia, Ghana, Kenya$\}$.