10

Syntax and Semantics

10.1 Validity

Logic is to do with *arguments.*

An argument starts with some statements—the *premises*—and then takes us via a series of steps to another statement—the *conclusion.*

Arguments are designed to expand our knowledge. If you already know the premises, then a good argument will lead you to knowledge of the conclusion too.

Given this function, what we want of an argument is that the truth of its premises should *guarantee* the truth of its conclusion. An argument satisfying this desideratum is called *valid.*

Note that the validity of an argument doesn't require that the premises and conclusion actually be true—only that the conclusion must be true *if* the premises are true.

For example, consider this argument:

All Australians like cricket
Mel Gibson is Australian

Mel Gibson likes cricket

Now, you might wonder whether the premises and conclusion of this argument are in fact true. But you don't need to settle this to know that the argument is valid. You can see that the premises guarantee the conclusion all right—in that the conclusion would be true *if* the premises were—independently of whether these statements actually are true.

It is not the job of an argument, so to speak, to check that its premises are true. That comes from outside the argument. The argument is solely concerned with the move *from* the premises *to* the conclusion, and it will have played its part as long as the truth of the former guarantees the truth of the latter.

10.2 Logic and Metalogic

We can regard logic as a skill, something we can be better or worse at. In this sense good logicians are people who are sensitive to the difference between valid arguments and invalid ones, and who go in for valid argumentation themselves. Some elementary logic courses are designed to improve this kind of skill. They aim to turn their students into valid arguers.

So construed, logic has no special subject matter. It is a generic skill that can be used—and should be used—in any area of thought. It is good for engineers and lawyers to be good logicians in this sense, as well as philosophers.

But we can also regard logic as an object of study. We can think *about* different ways of arguing validly, and analyse their workings. When we do this we are doing *metalogic*.

Metalogic, unlike logic, has a quite specific subject matter—the workings of logical arguments. Metalogic is of great interest to philosophers and mathematicians, but not necessarily to lawyers and chemists.

When philosophers and mathematicians talk about a 'good logician', they are likely to mean someone who knows a lot of metalogic—knows a lot *about* logical arguments—and not just someone who is good at arguing logically.

This chapter and the next two will contain some metalogic. This won't be designed to improve your argumentative skills. Rather my aim will be to introduce you to some philosophically interesting facts about logical arguments.

10.3 Different Kinds of Logic

We can classify logical arguments according to the way their validity depends on the meaning of certain logical constructions.

In studying *propositional* logic, we are concerned with arguments whose validity depends on the meanings of the '*truth-functional connectives*'—'not', 'and', 'or', and so on.

In studying *predicate* logic, we are further concerned with arguments whose validity depends on the meanings of the '*quantifiers*'—'for all x, . . . ' and 'there is an x such that, . . . '.

Other branches of logic, such as 'second-order logic' and 'modal logic', involve arguments whose validity depends on the meaning of yet further constructions.

10.4 Truth-Functional Connectives

Let us stick to propositional logic for the moment. We shall consider some of the other branches of logic in the next chapter.

The 'truth-functional connectives' of propositional logic are specifically those words that can be used to make new sentences out of old ones in such a way that the truth or falsity of the new sentences is entirely determined by the truth and falsity of the old ones.

So given any sentence p, *'not'-p* can be defined as a new sentence which is true if and only if p is false.

Similarly, given any two sentences p, q, *p-'and'-q* can be defined as a new sentence which is true if and only if p is true and q is true.

Again, *p-'or'-q* (with 'or' understood as 'and/or') can be defined as a new sentence which is true if and only if at least one of p and q is true.

And I will say that *p-'→'-q* can be defined as a new sentence which is true if and only if either p is false or q is true. (This is the material conditional discussed at the end of chapter 8. I'll come back to this one in a second.)

I have given these definitions in words, but they can be made graphic by the '*truth tables*' that will be familiar to anyone who has done an elementary logic course. These truth tables illustrate directly how the truth values of the relevant complex sentences depend on the truth values of their constituents. (See Box 20.)

'Not', 'and', 'or', and '→' aren't the only truth-functional connectives, but they are enough for our purposes.

I have just offered '*definitions*' of the words 'not', 'and', and so on. But of course it's not up to me to *choose* their meanings. They are already words of English with a life of their own, so to speak. So it is a substantial question whether the definitions I have offered are faithful to the meanings they already have.

And indeed there are respects in which the relevant English words do have connotations which go beyond the above definitions. Still, it will not hurt to ignore that here in the interests of simplicity.

The one exception is with the connective I have written as p-'→'-q. It is normal in introductions to logic to equate this with the English construction 'if p, then q'. But, as I explained in chapter 8, there is a quite substantial divergence between the logicians' p-'→'-q and the ordinary language 'if p, then q'. In recognition of this, I shall stick to the artificial '→' in this chapter.

Box 20 Truth Tables

The truth tables for 'not', 'and', 'or', and '→' show graphically how the truth values of the complex sentences we can make using these words depend on the truth values of their constituent sentences.

p	'not'-p
T	F
F	T

p	q	p-'and'-q
T	T	T
T	F	F
F	T	F
F	F	F

p	q	p-'or'-q
T	T	T
T	F	T
F	T	T
F	F	F

p	q	p-'→'-q
T	T	T
T	F	F
F	T	T
F	F	T

10.5 Syntax and Semantics

A central aim of metalogic is to construct a precise analysis of *logical consequence*—the relationship that some sentences (the premises) have to another (the conclusion) when the latter follows validly from the former.

One thing that makes metalogic interesting is that there are two different ways of thinking about logical consequence—*syntactically* and *semantically*.

When we analyse logical consequence *syntactically*, we think of argumentation as governed by a system of rules for moving between sentences of certain forms. One sentence is the logical consequence of some others if the rules allow you to construct a 'proof' in the sense of a sequence of legitimate moves that take you from the premises to the conclusion. From this syntactic perspective, the meanings of the sentences do not matter. Argumentation is viewed as nothing more than a *game* governed by certain *rules* for manipulating strings of symbols.

The *semantic* perspective, by contrast, attends to meanings rather than moves. Now we think of sentences not just as strings of symbols, but as meaningful statements which make claims that are true or false. And this allows us to view one sentence as a logical consequence of others just in case their meanings are so related that the former sentence must be true if the latter are. From the semantic perspective, logical consequence is nothing to do with argumentative *moves*. It's simply a matter of all circumstances in which the premises are true being ones in which the conclusion is true too.

In the next two sections I shall illustrate these two different ways of understanding logical consequence in the case of propositional logic.

10.6 Syntactic Consequence

So let us first view propositional logic syntactically, as a game with certain rules. The most natural way to specify the rules is to list a set of *rules of inference*. Each rule of inference will allow you to move from sentences of certain forms to another sentence of a related form.

For example, here are two nice simple rules:

Given any two sentences p, q, you can move to p-'and'-q
Given any sentence of the form p-'and'-q, you can move to p (and similarly to q)

We can give a similar pair of rules for each of our other truth-functional connectives, the first of which allow us to move *from* sentences without the connective *to* sentences containing it (an '*introduction*' rule) and the other which allows us to move *from* sentences containing the connective *to* sentences without it (an '*elimination*' rule). (See Box 21.)

Once we have specified a set of rules of inference, we can then define a *proof*. A proof is a way of moving by steps from a set of premises to a conclusion using the rules of inference. More formally, a *proof* consists of an initial set of sentences given as *premises*, followed by a sequence of sentences each of which can be reached by the rules of inference from the premises plus other sentences earlier in the sequence. The last sentence in such a sequence is the *conclusion*. (See Box 22.)

This now gives us enough to define a syntactic notion of logical consequence for propositional logic. A sentence j is a syntactic consequence of a set of sentences K in propositional logic if there is a proof with premises K and conclusion j.

In such a case we write $K \vdash_{PROP} j$, and we say that j is *provable* from K in propositional logic.

If we can prove j from zero premises, we write $\vdash_{PROP} j$ and say that j is *provable* simpliciter, or that j is a *theorem* of propositional logic.

(If you are puzzled about how anything can be proved from zero premises, have a look at the rule of Conditional Proof in the Box 21 below.)

Box 21 Inference Rules for Propositional Logic

'And' Introduction

Given p, q, move to p-'and'-q

'And' Elimination

Given p-'and'-q, move to p (or to q)

'Or' Introduction

Given p (or given q), move to p-'or'-q

'Or' Elimination

Given p-'or'-q, p-'→'-r , q-'→'-r, move to 'r'

Reductio Ad Absurdum ('Not' Introduction)

Given p-'→'-q, p-'→'-not-q, move to 'not'-p

Double 'Not' Elimination

Given 'not'-'not'-p, move to p.

Modus Ponens ('→' Elimination)

Given p, p-'→'-q, move to q.

Conditional Proof ('→' Introduction)

If assuming p allows you to move via this set of Inferential Rules to q, then you can move to p-'→'-q without assuming p.

10.7 Semantic Consequence

Note how nothing in the syntactic approach to propositional logic just outlined appeals to the meanings of sentences. From the syntactic perspective, the sentences may just as well be meaningless marks, and the inference rules may as well just specify the allowed moves in some arbitrary game.

But of course sentences aren't just meaningless marks—they express propositions which are true or false. To take this into account is to view the sentences *semantically*.

I have already explained, in section 10.4 above, how the truth-functional connectives can be viewed as devices which function to generate complex sentences whose *truth values* (their truth or falsity) are *determined* by the *truth values* of their constituents. This gives us a *semantic* understanding of sentences involving truth-functional connectives. We see how the truth values of these sentences depend on the truth values of their simpler parts.

Box 22 An Example of a Syntactic Proof

Here is a proof of 'not'-p from p-'→'-q and 'not'-q.

(1) Premise p-'→'-q
(2) Premise 'not'-q

Suppose we now *assume* p
We were given 'not'-q as a premise
So *assuming* that *p* allows us to move to 'not'-q.
So without assuming *p* we can move, via **'→'** *Introduction*, to

(3) p-'→'-'not' -q

And from (1) and (3) we can move to

(4) 'not'-p by 'Not' Introduction.

Once we are armed with this semantic grasp of the truth-functional connectives, we can approach the issue of logical consequence from a semantic rather than a syntactic perspective. Instead of asking whether we can *move* from some sentences to another via the specified rules of inferences, we can simply ask whether their meanings are so related that the latter *must be true* if the former *are true*.

So, for example, suppose we are interested in whether some sentence 'not'-p is a logical consequence of p-'→'-q and 'not'-q. Then we can easily see, by attending to the relevant truth tables, that the conclusion must indeed be true if the premises are. (See Box 23.)

This illustrates the semantic notion of logical consequence for propositional logic. A sentence j is a semantic consequence of a set of sentences K in propositional logic just in case the definitions of the truth-functional connectives ensure that j must be true whenever the sentences in K are all true.

In such cases we write K $\models_{PROP}$ j, and we say that j is a *semantic* consequence of K in propositional logic.

If j must be true whatever is the case, then we write $\models_{PROP}$ j, and we say that j is a propositional *logical truth*.

For example any sentence of the form p-'or'-'not'-p is a propositional logical truth. The semantic definitions of 'or' and 'not' ensure that any such sentence is true whatever p says.

Now that we have explained and contrasted the syntactic and semantic notions of logical consequence, we can ask about their relationship to each other. That will be the subject of the next chapter.

Box 23 An Example of Semantic Consequence

The following truth table shows that 'not'-p is a semantic consequence of p-'→'-q and 'not'-q: the bottom row represents the only case where p-'→'-q and 'not'-q are both true—and in that row 'not'-p is also true

p	q	p-'→'-q	'not'-q	'not'-p
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

FURTHER READING

There are many excellent elementary logic textbooks, including Wilfried Hodges' *Logic* (Penguin 2nd edition 2001) and Paul Tomassi's *Logic* (Routledge 1999).

The elementary book that pays most attention to metalogical issues is still the classic *Beginning Logic* by E.J. Lemmon (Nelson 1965).

The Logic Manual by Volker Halbach (Oxford University Press 2010) also has a usefully metalogical slant.

EXERCISES

1. Give examples of valid arguments with:

   (a) true premises and true conclusion
   (b) false premises and false conclusion
   (c) false premises and true conclusion

   Why haven't I asked for an example with true premises and false conclusion?

2. Use truth tables to show that the following are logically true: p-'or'-'not'-p; 'not'-(p-'and'-'not'-p); (p-'and'-'not'-p-)'→'-q.

3. Which of the following claims are true? (Indicate your reasons for your answer.)

   (a) 'not'-(p-'and'-'not'-q), 'not'-p $\models$ q
   (b) $\models$ ((p-'or'-'not'-q)-'and'-q)-'→'-p
   (c) $\vdash$ p-'→'-(p-'or'-q)
   (d) $\vdash$ p-'→'-(p-'and'-q)

4. State three rules of inference from propositional logic and use the truth tables for the connectives involved to show that their conclusions must be true if their premises are.

5. Use the definitions of K $\models_{PROP}$ q and $\models_{PROP}$ q and the truth table for '→' to explain why: p $\models_{PROP}$ q if and only if $\models_{PROP}$ p → q.