# Philosophical Devices:

## Proofs, Probabilities, Possibilities, and Sets

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# 2

# Infinite Sets

## 2.1 Some Infinite Sets

Some sets have infinitely many members.

Think of the set of all the New Year's Eves from here to eternity.

Or if you don't believe in eternity, think of the set of all the spatial points between London and New York. (Since there will always be another point between any two distinct such points, there will be no end of them.)

Again, think of the set of all grammatical English sentences. (Since there is no word limit on the length of English sentences, we can always go on making longer sentences from shorter ones by such devices as adding 'John said that' to the beginning, or putting 'and then they had tea.' at the end.)

These are slightly messy examples. If you want a nice clean example of an infinite set, simply take the set of all the natural numbers, $\{0, 1, 2, 3, \ldots\}$.

While we are on numbers, take care not to confuse numbers with the *numerals* that name them. (See Boxes 4 and 5.) Numerals are *words* like 'one' and 'two' or *symbols* like '1' and '2'. Numbers are the more abstract things that these numerals name. The English word 'two' is a

**Box 4** Use and Mention

Philosophers are very fussy (because they often need to be) about distinguishing words from the things that they refer to. If you want to talk about the word rather than the thing, you must put the word in quotes to form a name of that word itself. Here are some examples that illustrate this device.

> London contains ten million people, but 'London' contains six letters.
> Jack is an unpopular person, but 'Jack' is a popular name.
> Seven is an odd number, but 'seven' is an English word—a numeral.
> {John, Paul, George, Ringo} is the same set as {x: x is a Beatle}, but '{John, Paul, George, Ringo}' and '{x: x is a Beatle}' are two different names for that set.

On the left-hand side of these examples we *use* the names, on the right we *mention* them.

different word from the French word 'deux', but they both name the same number. Again, the Arabic '2' is a different symbol from the Roman 'II' but they also both name the same number. Numerals are signs used in specific representation systems. Numbers themselves are timeless entities that transcend the perspective of any given system of representation. (See Box 6.)

## 2.2 Different Kinds of Numbers

The most basic numbers are the *natural numbers*: 0, 1, 2, 3, …

If we add the negative whole numbers to the natural numbers, then we get the *integers*: …-3, -2, -1, 0, 1, 2, 3…

In addition to the integers, we also want to recognize various kinds of intermediate numbers, numbers that fall between the integers.

The simplest are the *rational numbers,* namely those that can be expressed as fractions of the form p/q, where p and q are integers.

But we also need to recognize further numbers that are not rational.

For example, $\sqrt{2}$ is not rational. There is no way to express $\sqrt{2}$ in the form p/q where p and q are integers. (See Box 7.)

Similarly, π (the ratio of a circle's circumference to its diameter) is not rational. It cannot be expressed as p/q with integral p and q either.

Many other numbers are similarly irrational.

The *real* numbers comprise both the rational and irrational numbers.

Any real number can be represented by an infinitely long decimal expansion: e.g. 23.17564839…

**Box 5** Types and Tokens

cat cat

*Question*. How many words were there in the previous line? *Answer.* One word *type*, but two *tokens* of that type.

The term " 'cat' " can refer either to the type word or to some specific token of it.

Thus: *'cat' occurs often in children's stories*. Here I use " 'cat' " to refer to a word type.

But now consider: *the first 'cat' at the beginning of this Box could have been written with a capital letter*. Here I use " 'cat' " to refer to a specific token of the relevant type.

(Note how I have to use double quotes—" 'cat' "—to *mention* the *name* of the original word, that is, the name that we formed by putting that original word in single quotes.)

**Box 6** The Reality of Numbers

As with sets, it is possible to doubt whether numbers really exist. If they are outside space and time, and have no causal impact on anything, do we really need to believe in them? Some philosophers are indeed inclined to dismiss numbers, along with sets, as no more than useful fictions. But, as before, we can bypass this issue here, and think of ourselves as exploring what properties numbers *would* have, *if* they existed. Even those who are suspicious of numbers will do well to understand their workings, so to speak.

In this format, we can distinguish the rational numbers from the irrational ones by the fact that the rational numbers will eventually display some recurring sequence of digits. So for example, 1/11 is 0.090909... and 2/7 is 0.285714285714285714... (See the Exercises for some hints about how to show that the rational numbers are just those whose decimal expansions recur.)

## 2.3 Two Senses of 'More'

Here is a good question. Are there more natural numbers than even numbers?

In one obvious sense the answer must be yes. The set of even numbers {0, 2, 4, 6,...} is a *proper subset* of the set of natural numbers {0, 1, 2, 3, ...}. The latter set contains all the members of the former set and then some. There are plenty of natural numbers that aren't even, but no even numbers that aren't natural.

**Box 7** $\sqrt{2}$ is Irrational

Suppose (for the sake of a 'reductio ad absurdum' proof) that $\sqrt{2}$ *is* rational and so can be represented as p/q, where p and q are integers, and suppose further that p and q have no common factors, that is, that all cancelling has been done. Then it follows:

$$\sqrt{2} = p/q$$
$$2 = p^2/q^2$$
$$2q^2 = p^2$$

So p must be an even number (since its square is an even number). So, for some integer r, p must be 2r. So

$$p^2 = 4r^2$$

And, since we already know that $2q^2 = p^2$, it follows that

$$q^2 = 2r^2$$

So q must be an even number too. But now q and p are both even, which contradicts the supposition that $\sqrt{2}$ is rational and represented as p/q with no common factors. So by reductio we can conclude that $\sqrt{2}$ is irrational.

When the Greeks first discovered that $\sqrt{2}$ is irrational, it freaked them out. They knew from Pythagoras' theorem that $\sqrt{2}$ is the length of the hypotenuse of a right-angled triangle whose other sides are each of length 1. But the irrationality of $\sqrt{2}$ means that there can be no unit of length that will fit exactly q times into these short sides and p times into the hypotenuse (for if there were, then $\sqrt{2}$ would equal p/q). To the Greeks, this seemed to contradict the very idea of length. It is said that the Greek mathematicians who first proved the irrationality of $\sqrt{2}$ tried to keep their discovery a secret.

But in a different sense the answer is no. The even numbers can be *paired up one-to-one* with the naturals. In this sense there are just as many even numbers as natural numbers.

| | | | | | |
|---|---|---|---|---|---|
| 0 | 2 | 4 | 6 | 8 | ... |
| 0 | 1 | 2 | 3 | 4 | ... |

This mapping gives a unique even number for every natural number, and vice versa.

There is no contradiction here. We can distinguish two senses in which set A can contain 'more members' than set B. In the first sense (call it the 'subset' sense), it simply means that B is a proper subset of A. In the second sense (the 'pairing' sense), it means rather that any attempt to pair the members of A one-to-one with those of B will leave some members of A unpaired.

There are more natural numbers than even numbers in the subset sense, but not in the pairing sense—for the pairing illustrated above succeeds in matching every natural number with its own even number.

When we are dealing with finite sets, the two senses of 'more' coincide. If a finite set B is a proper subset of finite set A, then the As can't all be paired up one-to-one with the Bs, for there won't be enough Bs—any attempted pairing will leave some extra As unpaired.

But with infinite sets, B can be a proper subset of A, and still be paired up one-to-one with the As—for now the Bs won't automatically run out before we get to the end of the As.

This is in fact a defining characteristic of infinite sets. The members of any infinite set, but of no finite set, can be paired up one-to-one with the members of some of its proper subsets.

## 2.4 Denumerability

The odd numbers can also be paired one-to-one with the natural numbers.

| | | | | | |
|---|---|---|---|---|---|
| 1 | 3 | 5 | 7 | 9 | ... |
| 0 | 1 | 2 | 3 | 4 | ... |

So can the squared whole numbers.

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 4 | 9 | 16 | ... |
| 0 | 1 | 2 | 3 | 4 | ... |

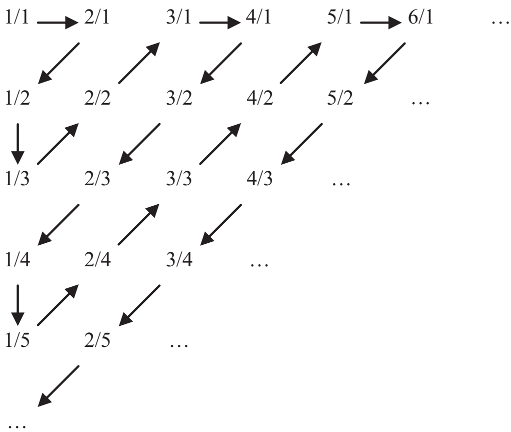
And all the integers.

| | | | | | |
|---|---|---|---|---|---|
| 0 | -1 | +1 | -2 | +2 | ... |
| 0 | 1 | 2 | 3 | 4 | ... |

What about the rational numbers? At first sight it might seem that there are too many. There really are an awful lot. In particular, given any two rational numbers, however close together, there will always be another rational number in between them. (Mathematicians call this property 'density'.) You might think that this would block any attempt to line them up with the natural numbers.

Surprisingly, however, the rational numbers can also be paired up one-to-one with the natural numbers. To see this, consider the following grid. It clearly contains all the rational numbers. And the arrows indicate a systematic way of going through the grid in sequence and thereby placing the rational numbers in a numerical order.[1]

[1] A little complication. If we list the rational numbers as in the diagram below, any given rational number will recur in different guises at different points in the list. For example, we will not only have 1/2, but later on 2/4, 3/6, and so on. Since these are all the same rational number, just written in different ways, our list won't really pair each rational number with a *unique* natural number. The remedy is to complicate the listing procedure a bit—before writing down the $n^{th}$ rational number, check that it hasn't already occurred in the list, and throw it away if it has.




Whenever the members of a set can be paired one-to-one with the natural numbers, we say the set is *denumerable*. A denumerable set is one that can be placed in a numerical list. A numerical list, if you think about it, just *is* a pairing of the listed items with the natural numbers—the first in the list with 1, the second with 2, and so on.

## 2.5 More Denumerable Sets

Many unruly-looking sets can be shown to be denumerable.

Take the set of all rectangles with rational length and breadth, for example. Each of these is defined by two rational numbers. Given that we can place all the rational numbers themselves in a numerical list, by the grid trick above, we can thus equate each of these pairs of rational numbers with a pair of *natural* numbers. And then we can apply the grid technique once more, to place these pairs of natural numbers themselves in a numerical list. This will then amount to a numerical list of the rectangles we started with.

Or take the set of all English sentences. To place these in a numerical list, consider all finitely long strings of English letters (counting a space as a $27^{th}$ letter). Now order the one-letter strings alphabetically, then the two-letter strings, and so on. Now go through the resulting list and throw away all the strings which don't make sense as English sentences. You'll be left with a numerical list of English sentences.

There are many similar examples of denumerable sets.

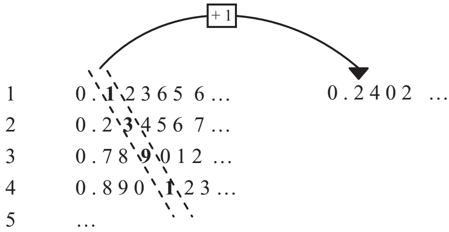
## 2.6 The Non-Denumerability of the Real Numbers

We have just seen that many complicated-looking infinite sets turn out to be denumerable. Does this hold for all infinite sets? Our surprising success at pairing the rational numbers and other unpromising-looking sets with the natural numbers might make you think that a similar trick can be pulled with all infinite sets. But that would be a mistake. The *real numbers* cannot be paired one-to-one with the natural numbers. They are *non*-denumerable. Indeed the reals between 0 and 1, or in any finite interval, are non-denumerable.

To show this, suppose (for the sake of another reductio argument) that the reals between 0 and 1 *were* denumerable. Then they could be paired up with the natural numbers in some way. To illustrate, suppose the pairing starts as in the list below. (This is just for illustration—the argument will work whatever the pairing.)

| | |
|---|---|
| 1 | 0.123456… |
| 2 | 0.234567… |
| 3 | 0.987654… |
| 4 | 0.976543… |

Now construct a new number according to the following rule: make the first digit one more than the first digit of the first number in this list, the second digit one more than the second digit of the second number, the third digit one more than the third digit of the third number, and so on…(using 0 as 'one more than 9' whenever the $n^{th}$ digit in the $n^{th}$ number is 9).




So, given our supposed initial listing of the reals, our new number will be 0.2402…And note that this new number *can't be anywhere in the original list,* since it differs from the first number in the first digit, from the second in the second digit, and so on.[2]

This is Cantor's famous diagonal argument. It shows that there are more real numbers than natural numbers *even in the one-to-one pairing*

[2] There is a little complication in this diagonal proof too. Some real numbers have two decimal representations. Consider for example 0.999… = 3 x 0.333… = 3 x 1/3 = 1. This shows that 0.999…and 1…are the same real number written in different ways. And this might make you worry that Cantor's argument only proves that there is a 'diagonal *representation*' that isn't in the original list of decimal *representations,* not that there is a real *number* that isn't among the *numbers* named by that list—for maybe the 'diagonal representation' is just an alternative name for one of the numbers already listed.

Well, it would be interesting enough to know that the set of decimal representations is itself non-denumerable, even if the real numbers themselves aren't. But in any case it is easy enough to tighten the proof so as to plug this hole. One of the Exercises at the end of Chapter 3 covers this.

*sense* of 'more'. If you try to pair up the reals with the naturals you will always have some real number left over. Given any supposed listing of the reals, it is always possible to construct another real number that isn't in that list.

## 2.7 The Abundance of the Real Numbers

The reals are very abundant indeed. To get some feel for this, recall that the real numbers are represented by *infinitely* long decimal strings, including strings that display no recurring patterns. The other entities we have been dealing with (rational numbers, sentences, ...) can all be represented in finite terms. This doesn't stop there being infinitely many rational numbers or sentences—finite representations can get longer and longer. But once we switch to *infinitely* long strings of digits, we are dealing with a quite different order of plurality.

The example of the reals shows that infinite sets come in different sizes. There is the size shared by all the denumerable sets. But the real numbers are bigger again. In the next chapter we shall explore the way in which different infinite sets can have different sizes in this way.

## FURTHER READING

*Numbers: A Very Short Introduction* by Peter Higgins (Oxford University Press 2011) explains the different kinds of numbers.

The last two chapters of Eric Steinhart's *More Precisely: The Math You Need To Do Philosophy* (Broadview Press 2009) deal with infinite sets and the variety of infinite numbers.

*An Introduction to the Philosophy of Mathematics* by Mark Colyvan (Cambridge University Press 2012) is a short and punchy introduction to the philosophical issues raised by numbers and mathematical objects.

James Robert Brown's *Philosophy of Mathematics: An Introduction to a World of Proofs and Pictures* (Routledge 1999) is another lively introduction to this area.

## EXERCISES

1. Write a sentence that both uses and mentions the word 'philosophy'. Write a sentence that both uses and mentions some other word. Say where in the two sentences the relevant words are used and where mentioned.

2. *7 7*
   How many token numerals are on the previous line? How many type numerals?
   How many natural numbers are less than 10? How many Arabic type numerals are written with one digit?

3. Show how all the integral multiples of 5 (positive and negative) can be paired one-to-one with the natural numbers.

4. Which of the following are subsets of the natural numbers?

   (a) the squares of the natural numbers
   (b) the square roots of the natural numbers

(c) the positive whole numbers less than 10 million
(d) the rational numbers

5*. Show that any rational number p/q, with p and q integers, will have a decimal expansion that eventually recurs. (Hint: think about what will happen as you generate the decimal expansion by dividing q into p.)

6*. Show that any decimal number that terminates with a recurring part is equal to some rational number. (Hint: first separate the recurring part, then multiply it by $10^k$, where k is the number of digits in the recurring part, then see what happens when you subtract the original recurring part from this number.)

(*Exercises with starred numerals are more difficult.)

# 3

# Orders of Infinity

## 3.1 Some Harder Stuff

This chapter will be a bit harder.

I regard the issues covered so far as something every educated person should know about. (Maybe my expectations are a bit high—but you get the idea.)

The subject matter of this chapter, however, will be rather more esoteric. I shall explain some points relating to different kinds of infinities. This is not the kind of thing that is normally covered in an introductory philosophy book.

Still, it seems a pity not to go a bit further, now that we have come this far. The material in this chapter is philosophically intriguing, and easy enough to explain in the light of the last two chapters.

## 3.2 The Numerical Size of Sets

Let us start by thinking about the numerical size of sets, in the sense of how many members they have. (Mathematicians speak here of the '*cardinality*' of sets, but I shall stick to the more familiar 'numerical size'.)

In the last chapter we paid attention to ways in which the members of different sets can be paired up one-to-one. In effect, these pairing relationships determine the numerical size of sets. Two sets have the same *number* of members just in case their respective members can be paired up one-to-one.

This is obvious with finite sets. Two finite sets can be paired up one-to-one if and only if they have the same number of members. Indeed we can think of the natural numbers precisely as ways of characterizing the pairing properties of finite sets. Suppose we group the finite sets by whether their members can be paired up one-to-one. So first we have the empty set, then all the sets with a single member, then all the sets with a pair of members, and so on. We can then think of the natural numbers—0, 1, 2,...—as entities which characterize the common numerical size of the sets in each of these groups. So the number 0 represents the size of the empty set, the number 1 the size of all the single-membered sets, the number 2 the size of all the sets with a pair of members,..., the number 8 the size of all the eight-membered sets, and so on.

Now let us extend this kind of thinking to infinite sets. Suppose we group the infinite sets by seeing whether their members can be paired up one-to-one. So all the denumerable sets will be in one group, for example, and all the sets that can be paired with the real numbers between 0 and 1 in another. Then we can think of all the sets in such a grouping as having the same number of members. So there will be one 'infinite number' that characterizes the denumerable sets, and a distinct and bigger 'infinite number' that characterizes the real numbers between 0 and 1.

If asked, most people would probably say that all infinite sets have the same number of members—infinitely many. What more is there to say about the size of sets which outrun any finite numbering? However, the non-denumerability of the reals has shown us that this reaction is too quick. Given that the real numbers between 0 and 1 cannot be paired up with the natural numbers, we have no choice but to recognize at least two infinite numbers. There is the infinite number that characterizes the denumerable sets, and the distinct and bigger infinite number that characterizes all the sets whose members can be paired up with the real numbers.

In fact we shall see soon enough that there are many more infinite numbers than just these two. Once you start generating infinite numbers it is hard to stop.

## 3.3 The Reals and the Power Set of the Natural Numbers

It is not hard to show that the set of *real numbers between* 0 *and* 1 has the same numerical size as the set of *all subsets of the natural numbers* (the 'power set' of the natural numbers, in the terminology introduced in Chapter 1).

To see why, suppose we write the real numbers between 0 and 1 in binary notation—e.g. 0.*1100101*... (Binary notation is simply an alternative way of representing numbers, using powers of 2 where our familiar decimal notation uses powers of 10. See Box 8.) Then we can view each real number as a *recipe* for constructing a subset of the natural numbers: put 0 in the subset just in case there is a '*1*' in the first digit of the binary expression; put 1 in the subset just in case there is a '*1*' in the second digit of the binary expression; ... put n in the subset just in case there is a '*1*' in the $(n+1)^{th}$ digit of the binary expression; ...

This construction demonstrates that each real number between 0 and 1 can be taken uniquely to determine a subset of the natural numbers. And similarly each subset of the natural numbers uniquely determines a real number between 0 and 1 (... put a '*1*' for the $(n+1)^{th}$ digit of the binary expression just in case n is in the subset ...). (See Box 8.)

**Box 8** The Real Numbers and the Power Set of the Natural Numbers

Ordinary decimal notation represents numbers as sums of multiples of powers of 10. So for example:

$$107.25 = (1 \times 10^{2}) + (0 \times 10^{1}) + (7 \times 10^{0}) + (2 \times 10^{-1}) + (5 \times 10^{-2})$$

Binary notation does the same thing but uses powers of 2 in place of powers of 10. So for example in binary notation the decimally represented 107.25 comes out as:

$$1101011.01 = (1 \times 2^{6}) + (1 \times 2^{5}) + (0 \times 2^{4}) + (1 \times 2^{3}) + (0 \times 2^{2}) + (1 \times 2^{1}) + (1 \times 2^{0}) + (0 \times 2^{-1}) + (1 \times 2^{-2}) = 64 + 32 + 8 + 2 + 1 + 1/4$$

Note how binary numerals are always strings of nothing but '*1*'s and '*0*'s (since multiplying by 2 moves you to the next higher power of 2).
So any real number between 0 and 1 can be represented as a (possibly infinite) string of '*1*'s and '*0*'s, for example:

*0.100111001010*…

And this string can then be used as a recipe for constructing a subset of the natural numbers, by including a natural number in the subset iff its matching binary digit is a '*1*':

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| The natural numbers: | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8… |
| Our binary string: | *1* | *0* | *0* | *1* | *1* | *1* | *0* | *0* | *1*… |
| The resulting subset: | {0, | | | 3, | 4, | 5, | | | 8…} |

Conversely, any subset of the natural numbers can be used as a recipe for constructing a binary numeral between 0 and 1, by putting '*1*'s in the binary string in just those places that correspond to numbers in the subset.

So the *real numbers between 0 and 1* and *the power set of the natural numbers* can be paired up one-to-one, and in this sense comprise sets of the same numerical size. (I shall drop the qualification 'between 0 and 1' henceforth, given that the set of *all* real numbers can be shown to have the same numerical size as the set of real numbers between 0 and 1. I leave this as an Exercise.)

Suppose we give the name '$\text{infinity}_0$' to the numerical size of the natural numbers and other denumerable sets, in recognition of the fact that this is the smallest of the infinite numbers.

Now recall that any finite set with n members has a power set with $2^n$ members—there are $2^n$ ways of making subsets if we have n members to play with.

Given this, it would seem natural to write the numerical size of the power set of the natural numbers as $2^{\text{infinity}_0}$.

And by this convention the numerical size of the real numbers will also be $2^{\text{infinity}_0}$, since they are the same numerical size as the power set of the natural numbers.[1]

[1] Don't worry too much about whether it makes sense to raise 2 to the power of $\text{infinity}_0$—that is, to multiply 2 by itself $\text{infinity}_0$ times. For our present purposes it will be enough to treat '$2^{\text{infinity}_0}$' as nothing more than a usefully mnemonic symbol for the numerical size of the power set of the natural numbers.

Still, for what it is worth, there is a natural way to do arithmetic with infinite numbers, and in this arithmetic we do find that:

$2 \times \text{infinity}_0 = \text{infinity}_0$ and
$\text{infinity}_0 \times \text{infinity}_0 = \text{infinity}_0$
*but*
$2^{\text{infinity}_0} > \text{infinity}_0$.

## 3.4 The Continuum Hypothesis

We know that $2^{\text{infinity}_0}$ is a distinct and bigger number than $\text{infinity}_0$. Where $2^{\text{infinity}_0}$ enumerates the real numbers, $\text{infinity}_0$ enumerates the natural numbers, and Cantor's diagonal argument showed us that the numerical size of the real numbers outruns that of the natural numbers.

But here is an interesting question. Is $2^{\text{infinity}_0}$ the *next biggest* infinite number after $\text{infinity}_0$?

There is no guarantee, if you think about it, that this should be so. Maybe there is a kind of infinite set which is intermediate in size between the natural numbers and the real numbers. This would be a set that is too big to be paired up with the natural numbers, but too small for all the real numbers to be paired up with it. If this were so, then the numerical size of this set would be an infinite number that came between $\text{infinity}_0$ and $2^{\text{infinity}_0}$.

Suppose we adopt the convention that '$\text{infinity}_1$' names the next biggest infinite number after $\text{infinity}_0$, '$\text{infinity}_2$' the next, and so on, for as long as we need to go on. (Mathematicians use '$\aleph_0$', '$\aleph_1$',... for this sequence—pronounced 'aleph-zero', 'aleph-one',... But let us stick to a convention that is easier to follow.)

Our question was whether $2^{\text{infinity}_0}$ is the next biggest infinite number after $\text{infinity}_0$. This can now be posed as the question of whether $2^{\text{infinity}_0}$ equals $\text{infinity}_1$, or whether it is a distinct and larger infinite number.

The claim that $2^{\text{infinity}_0}$ is the same as $\text{infinity}_1$ is the famous '*continuum hypothesis*'. (It is so-called because the real numbers—which are of size $2^{\text{infinity}_0}$, remember—are often thought of as representing a continuous arrangement of points along a line. The 'continuum hypothesis' is thus the hypothesis that the number of such points is the *next* largest infinite number after the number of the natural numbers.)

Amazingly, standard set theory fails to decide this question. Both the continuum hypothesis *and* its denial are consistent with the rest of set theory.

This is very strange. Standard set theory allows us to construct infinite sets as big as the natural numbers, and also ones as big as the real numbers. But it doesn't say whether or not there are any that are in-between in size.

I didn't go into any details at the end of Chapter 1 about the ways in which mathematicians have sought to improve on the failings of naive set theory. But they have devised a number of alternative axiomatic systems that aim to capture the essential features of sets. Yet none of these systems decides the continuum hypothesis. If sets really existed, you would expect there to be a fact of the matter here, and for axiomatic set theory to tell us what it is. The independence of the continuum hypothesis from axiomatic set theory adds weight to the philosophical case against the reality of sets.

(The discovery that the continuum hypothesis is left undecided by standard set theory came relatively late. In 1940 Kurt Gödel showed that the continuum hypothesis itself is consistent with set theory, and in 1963 Paul Cohen showed that the *denial* of this hypothesis is also consistent with set theory.)

## 3.5 An Infinity of Infinities

There is an infinity of different infinite numbers.

This follows from the fact that the power set of any set S is always of larger numerical size than the set S itself.

This 'power set theorem' can be proved by a generalized version of Cantor's diagonal argument. It shows that any attempt to pair the members of the power set of any set S with the members of S itself will inevitably omit some members of the power set. There are

**Box 9** The Power Set Theorem

Take any set S and its power set P(S). We want to show that there is no way of pairing the members of P(S) with those of S itself.

Suppose (for the sake of yet another reductio argument) that there is such a pairing.

| | | | | |
|---|---|---|---|---|
| Members of S: | *a* | *b* | *c* | … |
| Members of P(S): | L | M | N | … |

Now form a new subset K of S by going through all S's members one by one and sticking them in this new subset just in case they are *not* in the subset they are paired with. (So the new subset K contains *a* just in case *a* does *not* belong to L, and *b* just in case *b* does *not* belong to M, and so on.)

This new set K will now be a subset of our original S which is *different* from each of the subsets that were initially lined up with members of S.

To see that our constructed K must differ from each of the subsets that were initially lined up with members of S, note that K differs from L with respect to *a* (it contains *a* if and only if L doesn't contain it), and differs from M with respect to *b* (it contains *b* if and only if M doesn't contain it),…and in general differs from each of the subsets originally paired with the members of S with respect to just that member of S which that subset was originally paired with.

So we have derived a contradiction from the supposition that there is a way of pairing *all* the subsets of S with members of S itself. There can be no such pairing.

(If this reminds you of the 'diagonal' argument from the last chapter, so it should—we've just applied the same trick to subsets that we there applied to decimally represented numbers.)

always too many subsets of S to be paired with the members of S itself. (See Box 9.)

So, just as the power set of the natural numbers is bigger in size than the natural numbers themselves, so also is the power set of *that* set bigger again, and so on.

This guarantees that we have an infinite sequence of infinite numbers, each bigger than the one before. These numbers represent the numerical sizes of the sequence of sets generated from the natural numbers by repeatedly taking power sets.

In line with our earlier convention, it is natural to call these numbers '$\text{infinity}_0$', '$2^{\text{infinity}_0}$', '$2^{2^{\text{infinity}_0}}$', and so on. The rationale for this convention, as before, is that any set with n members has $2^n$ subsets.

## 3.6 The Generalized Continuum Hypothesis

To repeat, the sequence of numbers $\text{infinity}_0$, $2^{\text{infinity}_0}$, $2^{2^{\text{infinity}_0}}$,... enumerates the sequence of sets generated by repeatedly taking power sets of the natural numbers.

Now, analogously to our earlier question of whether or not $2^{\text{infinity}_0}$ is the same as $\text{infinity}_1$ (the continuum hypothesis), we can ask how this sequence of numbers $2^{\text{infinity}_0}$, $2^{2^{\text{infinity}_0}}$,... relates to the sequence $\text{infinity}_1$, $\text{infinity}_2$,... Remember that this latter sequence is simply the sequence of *all* infinite numbers after $\text{infinity}_0$ arranged in ascending order.

The '*generalized* continuum hypothesis' states that these two sequences coincide throughout. That is, the *generalized* continuum hypothesis asserts that the sequence $\text{infinity}_0$, $2^{\text{infinity}_0}$, $2^{2^{\text{infinity}_0}}$,... comprises all the infinite numbers. There are no infinite numerical sizes in between those generated by repeatedly taking power sets of the natural numbers. All infinite sets can be paired up one-to-one with one of the power sets generated in this way. (Compare the way that the simple continuum hypothesis said that $2^{infinity_0}$ coincided with $infinity_1$ and thus that there is no infinite numerical size in between those of the natural numbers and their power set.)

Again, the generalized continuum hypothesis isn't decided by the standard set theory.

## FURTHER READING

The last two chapters of Eric Steinhart's *More Precisely: The Math You Need To Do Philosophy* cover the material of this chapter in more detail.

*Set Theory and the Continuum Problem* by Raymond Smullyan and Mervyn Fitting (Dover revised edition 2010) goes deeper into a lot of the mathematics covered in this chapter.

Adrian Moore's *The Infinite* (Routledge 1990) deals with some of the philosophical issues raised by the notion of infinity.

## EXERCISES

1. (a) How many Arabic type numerals are there?
   (b) How many pairs of Arabic type numerals are there?
   (c) How many infinitely long strings of Arabic type numerals are there?

2. Suppose I have a numerical list of all the rational numbers in decimal representation. Why can't I use Cantor's diagonal argument to show that the rationals are non-denumerable?

3. Tighten Cantor's diagonal proof to deal with the problem of alternative decimal representations for the same real number. (Hint: we only get alternative decimal representations when one representation ends with infinitely many nines and the other with infinitely many zeros.)

4. In the text I said that the possibility of representing the real numbers between 0 and 1 in binary form demonstrates that each such number 'can be taken uniquely to determine a subset of the natural numbers'. But in fact this demonstration is not immediate. What is the complication?

5*. Show that all the real numbers can be paired one-to-one with the reals between 0 and 1.