Natural Deduction for Sentence Logic

6

Strategies

6–1. CONSTRUCTING CORRECT DERIVATIONS

Knowing the rules for constructing derivations is one thing. Being able to apply the rules successfully is another. There are no simple mechanical guidelines to tell you which rule to apply next, so constructing derivations is a matter of skill and ingenuity. Long derivations can be extremely difficult. (It's not hard to come up with problems which will stump your instructor!) At first, most students feel they don't even know how to get started. But with a bit of practice and experience, you will begin to develop some intuitive skill in knowing how to organize a derivation. To get you started, here are some examples and practical strategies.

Usually you will be setting a problem in the following form: You will be given some premises and a conclusion. You will be told to prove that the conclusion follows validly from the premises by constructing a derivation which begins with the given premises and which terminates with the given conclusion. So you already know how your derivation will begin and end.

Your job is to fill in the intermediate steps so that each line follows from previous lines by one of the rules. In filling in, you should look at both the beginning and the end of the derivation.

Let's illustrate this kind of thinking with a simple example. Suppose you are asked to derive 'B&C' from the premises 'A⊃B', 'A⊃C', and 'A'. Right off, you know that the derivation will take the form

1	A⊃B	*P*
2	A⊃C	*P*
3	A	*P*
	?	
	?	
	?	
	B&C	

where you still have to figure out what replaces the question marks.

First, look at the conclusion. It is a conjunction, which can most straightforwardly be introduced with the rule for &I. (From now on, I'm going to use the shorthand names of the rules.) What do you need to apply that rule? You need 'B' and you need 'C'. So if you can derive 'B' and 'C', you can apply &I to get 'B&C'. Can you derive 'B' and 'C'? Look at the premises. Can you get 'B' out of them? Yes, by applying ⊃E to lines 1 and 3. Similarly, you can derive 'C' from lines 2 and 3. Altogether, the derivation will look like this:

1	A⊃B	*P*
2	A⊃C	*P*
3	A	*P*
4	B	1, 3, ⊃E
5	C	2, 3, ⊃E
6	B&C	4, 5, &I

Let's try a slightly harder example. Suppose you are asked to derive 'C⊃A' from the premises 'AvB' and 'C⊃~B'. Your target conclusion is a conditional. Well, what rule allows you to conclude a conditional? ⊃I. So you will try to set things up so that you can apply ⊃I. This will involve starting a subderivation with 'C' as its assumption, in which you will try to derive 'A'. In outline, the derivation you are hoping to construct can be expected to look like this:

1	AvB		*P*
2	C⊃~B		*P*
3		C	*A*
		?	
		?	
		A	
	C⊃A		

(Your derivation won't **have** to look like this. In every case there is more than one correct derivation of a conclusion which follows from a given set of premises. But in this case, this is the obvious thing to try, and it provides the simplest correct derivation.)

To complete the derivation, you must fill in the steps in the subderivation to show that (given the premises of the outer derivation) 'A' follows from 'C'.

How will you do this? Let's study what you have available to use. In the subderivation you are allowed to use the subderivation's assumption and also any previous premise or conclusion in the outer derivation. Notice that from 'C' and the premise 'C⊃~B' you can get '~B' by ⊃E. Is that going to do any good? Yes, for you can then apply vE to '~B' and the premise 'AvB' to get the desired 'A'. All this is going to take place in the subderivation, so you will have to reiterate the premises. The completed derivation looks like this:

1	AvB		*P*
2	C⊃~B		*P*
3		C	*A*
4		C⊃~B	2, *R*
5		~B	3, 4, ⊃E
6		AvB	1, *R*
7		A	5, 6, vE
8	C⊃A		3–7, ⊃I

If you are still feeling a little lost and bewildered, reread the text from the beginning of this section.

When you have understood the examples given so far, you are ready for something new. Let's try to derive 'A⊃~B' from 'B⊃~A'. As in the second example, our first effort to derive a conditional should be by using ⊃I. So we want a subderivation with 'A' as assumption and '~B' as final conclusion:

```
1 | B⊃~A      P
  |----
2 |  | A      A
  |  |---
  |  |  ?
  |  |  ?
  |  | ~B
  | A⊃~B      ⊃I
```

But how can we get '~B' from the assumption of 'A', using the premise of the outer derivation?

'~B' is the negation of the sentence 'B'. Unless there is some really obvious simple alternative, one naturally tries to use ~I. ~I works by starting a subderivation with the sentence to be negated as assumption and then deriving some sentence and its negation. In the present situation this involves something that might not have occurred to you, namely, creating a subderivation of a subderivation. But that's fine. All the rules for working on a derivation apply to subderivations also, including the creation of subderivations. The only difference between a subderivation and a derivation is that a subderivation ends when we discharge its assumption, returning to its outer derivation; and that in a subderivation we may reiterate prior premises or conclusions from an outer derivation (or from any outer-outer derivation, as you will see in a moment). This is because in a subderivation we are working under the assumption that all outer assumptions and premises are true.

Will this strategy work? Before writing anything down, let me illustrate the informal thinking you should go through to see whether a strategy promises to be successful. Look back at the outline we have already written of how we hope the derivation will look. We are proposing to start a sub-sub-derivation with the new assumption 'B'. That sub-sub-derivation can use the original premise 'B⊃~A', which, together with the assumption 'B', will give '~A' by ⊃E. But the sub-sub-derivation is also within its outer derivation beginning with the assumption of 'A'. So 'A' is also being assumed in the sub-sub-derivation, which we express by reiterating 'A' in the sub-sub-derivation. The sub-sub-derivation now has both 'A' and '~A', which constitutes the contradiction we needed:


```
1 | B⊃~A              P
  |----
2 |  | A              A
  |  |---
3 |  |  | B           A
  |  |  |----
4 |  |  | B⊃~A        1, R
5 |  |  | ~A          3, 4, ⊃E
6 |  |  | A           2, R
7 |  | ~B             3-6, ~I
8 | A⊃~B              2-7, ⊃I
```

How are you doing? If you have had trouble following, rest for a moment, review to be sure you have gotten everything up to this point, and then we'll try something one step more involved.

Let's try deriving 'A≡~B' from 'AvB' and '~(A&B)'. The conclusion is a biconditional, and one derives a biconditional most easily by using ≡I. Think of a biconditional as the conjunction of two conditionals, the two conditionals we need to derive the biconditional using ≡I. So you should aim to develop a derivation along these lines:

```
1 | AvB        P
2 | ~(A&B)     P
  |----
  |   ?
  |   ?
  | ~B⊃A
  |   ?
  |   ?
  | A⊃~B
  | A≡~B       ≡I
```

We have reduced the complicated problem of deducing 'A≡~B' to the simpler problems of deducing '~B⊃A' and 'A⊃~B'.

In constructing derivations, you should learn to think in this kind of pattern. Try to resolve the problem of deriving the final conclusion (your target conclusion) by breaking it down into simpler problems of deriving simpler sentences (your new target conclusions). You may actually need to resolve your simpler problems into still more simple problems. You continue working backward toward the middle until you can see how to derive your simple sentences from the premises. At this point you start working from the premises forward and fill everything in.

How, in this example, can we derive our simplified new target conclusions? They are both conditionals, and as we saw in the second example, the straightforward way to derive conditionals uses ⊃I. This involves starting one subderivation for each of the conditionals to be derived:

```
1 | AvB          P
2 | ~(A&B)       P
  |-------
  |  | ~B        A
  |  |----
  |  | ?
  |  | ?
  |  | A
  | ~B⊃A         ⊃I
  |  | A         A
  |  |----
  |  | ?
  |  | ?
  |  | ~B
  | A⊃~B         ⊃I
  | A≡~B         ≡I
```

We have now resolved our task into the problem of filling in the two subderivations.

Can you see how to complete the subderivations by working with the premises of the outer derivation? The first subderivation is easy: '~B' and 'AvB' give 'A' by vE. The second subderivation presents more of a challenge. But we can complete it by using the same tactics illustrated in the previous example. We've assumed 'A' and want to get '~B'. To get '~B', we can try ~I (unless a really simple alternative suggests itself). ~I will require us to start a sub-sub-derivation with 'B' as assumption. In this sub-sub-derivation we can reiterate anything which is above in an outer derivation of the derivation on which we are working. So we can reiterate 'A',

which, with 'B', will give us 'A&B'; and we can reiterate the original premise '~(A&B)', thus giving us our contradiction. (Note that the contradiction can be in the form of any sentence and its negation. Neither sentence has to be atomic.) Since from 'B' we have derived a sentence and its negation, we can use ~I to discharge the assumption 'B', giving the conclusion '~B' at the end of the subderivation which began with 'A'. This is just what we needed.

If you find this tangle of thinking hard to unravel, read it over again, following each step in the completed derivation below to see how it all fits together.

```
 1 | AvB                  P
 2 | ~(A&B)               P
   |-------
 3 |  | ~B                A
   |  |----
 4 |  | AvB               1, R
 5 |  | A                 3, 4, vE
 6 | ~B⊃A                 3–5, ⊃I
 7 |  | A                 A
   |  |----
 8 |  |  | B              A
   |  |  |----
 9 |  |  | A              7, R
10 |  |  | A&B            8, 9, &I
11 |  |  | ~(A&B)         2, R
12 |  | ~B                8–11, ~I
13 | A⊃~B                 7–12, ⊃I
14 | A≡~B                 6, 13, ≡I
```

Now let's tackle something different. You are asked to derive 'C' from 'A&B' and '~C⊃~B'. What can you do? If you are stuck, you can at least write down the premises and conclusion so that you can start to see how the derivation will look:

```
1 | A&B      P
2 | ~C⊃~B    P
  |------
  | ?
  | ?
  | C
```

No rule applies immediately to the premises to give 'C'. Because 'C' is atomic, no introduction rule for a connective will give 'C'. What on earth can you do?

Sometimes when you are stuck, you can succeed by arranging to use ~I in what I am going to call the *Reductio Ad Absurdum* strategy. This strategy proceeds by assuming the negation of what you want and then from this assumption (and prior premises and conclusions) deriving a contradiction. As you will see in the example, you will then be able to apply ~I to derive the double negation of what you want, followed by ~E to get rid of the double negation. In outline, the reductio absurdum strategy, applied to this problem, will look like this:

```
1 | A&B        P
2 | ~C⊃~B      P
  |----
3 |  | ~C      A
  |  |----
  |  | X       ('X' here stands for some specific sentence, but I don't yet know
  |  | ~X      what it will be.)
  |
  | ~~C        ~I
  | C          ~E
```

Will this strategy work in this case? If you assume '~C', you will be able to use that assumption with the premise '~C⊃~B' to get '~B'. But '~B' will contradict the 'B' in the premise 'A&B', and you can dig 'B' out of 'A&B' with &E. In sum, from '~C' and the premises you will be able to derive both 'B' and '~B'. ~I then allows you to conclude '~~C' (the negation of the assumption which led to the contradiction). ~E finally gives 'C':

```
1 | A&B           P
2 | ~C⊃~B         P
  |----
3 |  | ~C         A
  |  |----
4 |  | ~C⊃~B      2, R
5 |  | ~B         3, 4, ⊃E
6 |  | A&B        1, R
7 |  | B          6, &E
8 | ~~C           3-7, ~I
9 | C             9, ~E
```


The first time you see an example like this it may seem tricky. But you will soon get the hang of it.

You do need to be a little cautious in grasping at the reductio strategy when you are stuck. Often, when students have no idea what to do, they assume the opposite of what they want to conclude and then start blindly applying rules. This almost never works. To use the reductio strategy successfully, you need to have a more specific plan. Ask yourself: "Can I, by assuming the opposite of what I want to derive, get a contradiction (a sentence and its negation) out of the assumption?" If you can see how to do this, you are all set, and you can start writing down your derivation. If you think you see a way which might work, it may be worth starting to write to clarify your ideas. But if you have no idea of how you are going to get a contradiction out of your assumption, go slow. Spend a little time brainstorming about how to get a contradiction. Then, if you find you are getting nowhere, you may well need to try an entirely different approach to the problem.

I should also comment on the connection between what I have called the reductio ad absurdum strategy and the rule for ~I. They really come to pretty much the same thing. If you need to derive a sentence of the form ~**X**, consider assuming **X**, trying to derive a contradiction, and applying ~I to get ~**X**. To derive a sentence of the form **X**, assume ~**X**, and derive ~~**X** by ~I. Then eliminate the double negation with ~E.

EXERCISES

6-1. For each of the following arguments, provide a derivation, complete with annotations, which shows the argument to be valid. If you find you are having difficulty with these problems, go over the examples earlier in this chapter and then try again.

```
a) Kv~I        b) ~C⊃A     c) ~D⊃~K     d) ~F⊃G
   ------         B⊃~A        K            G⊃~E
   ~(~K&I)        B           ~KvH         ------
                  ----        -----        E⊃F
                  C           D&H

e) A≡~B        f) AvB
   ------         B⊃C
   ~A⊃B           ~CvD
                  ----
                  ~D⊃A
```

6–2. RECOGNIZING THE MAIN CONNECTIVE

Suppose you are asked to provide a derivation which shows the following argument to be valid:

(1) (A⊃B)&[C≡(A⊃B)]
(2) ――――――――
    C

The premise is a mess! How do you determine which rule applies to it? After your labor with some of the exercises in the last chapter, you probably can see that the key lies in recognizing the main connective. Even if you got all of those exercises right, the issue is so important that it's worth going over from the beginning.

Let's present the issue more generally. When I stated the rules of inference, I expressed them in general terms, using boldface capital letters, '**X**' and '**Y**'. For example, the rule for &E is

(3) | [X&Y] / (X)   and | [X&Y] / (Y)

The idea is that whenever one encounters a sentence of the form **X&Y** in a derivation, one is licensed to write either the sentence **X** or the sentence **Y** (or both on separate lines) further down. Focus on the fact that this is so whatever sentences **X** and **Y** might be. This is the point of using boldface capital letters in the presentation. '**X**' and '**Y**' don't stand for any particular sentences. Rather, the idea is that if you write any sentence you want for '**X**' and any sentence you want for '**Y**', you will have a correct instance of the rule for &E. This is what I mean by saying that I have expressed the rule "in general terms" and by talking about a sentence "of the form **X&Y**".

How will these facts help you to deal with sentence (1)? Here's the technique you should use if a sentence such as (1) confuses you. Ask yourself: "How do I build this sentence up from its parts?" You will be particularly interested in the very last step in putting (1) together from its parts. In this last step you take the sentence

(4) A⊃B   which you can think of as **X**

and the sentence

(5) C≡(A⊃B)   which you can think of as **Y**

and put them on either side of an '&' to get the sentence

(A⊃B)&[C≡(A⊃B)]   which has the form **X&Y**

You have just established that (1) has the form of **X&Y**; that is, it is a conjunction with sentences (4) and (5) as its conjuncts. Consequently, you know that the rule for &E, (3), applies to sentence (1), so that if (1) appears in a derivation you are licensed to write sentence (4) or (5) (or both) below.

Similarly, if in a derivation you are faced with the sentence

(6) C≡(A⊃B)

ask yourself "What is the last thing I do to build this sentence up from its parts?" You take the sentence

(7) C   which you can think of as **X**

and the sentence

(8) A⊃B   which you can think of as **Y**

and you put them on either side of a biconditional, '≡', giving

(9) C≡(A⊃B)   which thus has the form **X≡Y**

Consequently, if you find sentence (6), you can apply the rule of inference for ≡E:

| [X≡Y] / (X⊃Y)   and | [X≡Y] / (Y⊃X)

which, when we put in sentences (7) and (8) for **X** and **Y**, look like

| C≡(A⊃B) / C⊃(A⊃B)   ≡E   and | C≡(A⊃B) / (A⊃B)⊃C   ≡E

Thus ≡E applies to sentence (6), licensing us to follow (6) on a derivation either with the sentence 'C⊃(A⊃B)', or the sentence '(A⊃B)⊃C', or both on separate lines.

In a moment we will apply what we have just done to provide a derivation which shows how (2) follows from (1). But we will need to see how to treat one more compound sentence. This time, try to figure it out for yourself. What is the form of the sentence '(A⊃B)⊃C'?

The last thing you do in putting this one together from its parts is to put '(A⊃B)' and 'C' on either side of a conditional, '⊃'. So the sentence has the form **X⊃Y,** with 'A⊃B' as **X** and 'C' as **Y.** If we have 'A⊃B' as well as '(A⊃B)⊃C' in a derivation, we can apply ⊃E to the two to derive 'C'.

Perhaps you can now see how we can write a derivation which shows (2) to follow from (1). In this case, because the desired objective, 'C', is atomic, we can't get it by an introduction rule. So it is useless to try to work backward from the end. The reductio ad absurdum strategy could be made to work, but only by doing more indirectly what we're going to have to do anyway. In this case the best strategy is to use elimination rules to take the premise apart into simpler pieces.

When we think through what these pieces will look like, we will see that they provide just what we need to derive 'C'. &E applies to the premise. '(A⊃B)&[C≡(A⊃B)]' to give us 'A⊃B' and 'C≡(A⊃B)'. In turn, ≡E applies to 'C≡(A⊃B)' to give us '(A⊃B)⊃C', which, together with the 'A⊃B', immediately gives us 'C' by ⊃E. (≡E applied to 'C≡(A⊃B)' also gives 'C⊃(A⊃B)'. But we have no use for 'C⊃(A⊃B)', so although licensed to write it down as a conclusion, we don't bother.) Altogether, the completed derivation looks like this:

1	(A⊃B)&[C≡(A⊃B)]	*P*
2	A⊃B	1, &E
3	C≡(A⊃B)	1, &E
4	(A⊃B)⊃C	3, ≡E
5	C	2, 4, ⊃E

The key idea you need to untangle a sentence such as (1) is that of a main connective:

> The *Main Connective* in a sentence is the connective which was used last in building up the sentence from its component or components.

(A negated sentence, such as '~(Av~B)', has just one component, 'Av~B' in this case. All other connectives use two components in forming a sentence.) Once you see the main connective, you will immediately spot the component or components to which it has been applied (so to speak, the **X** and the **Y**), and then you can easily determine which rules of inference apply to the sentence in question.

Let's practice with a few examples:

SENTENCE	MAIN CONNECTIVE	COMPONENT OR COMPONENTS
(AvB)⊃~(B&D)	⊃	AvB and ~(B&D)
[(A⊃B)v~D]v(~A≡D)	v	(A⊃B)v~D and ~A≡D
~[(A⊃B)⊃~(B⊃~A)]	~	(A⊃B)⊃~(B⊃~A)

The second and third examples illustrate another fact to which you must pay attention. In the second example, the main connective is a 'v'. But which occurrence of 'v'? Notice that the sentence uses two 'v's, and not both occurrences count as the main connective! Clearly, it is the second occurrence, the one used to put together the components '(A⊃B)v~D' and '~A≡D', to which we must pay attention. Strictly speaking, it is an occurrence of a connective which counts as the main connective. It is the occurrence last used in putting the sentence together from its parts. In the third example, '~' occurs three times! Which occurrence counts as the main connective? The very first.

In the following exercises you will practice picking out the main connective of a sentence and determining which rule of inference applies. But let's pause first to say more generally how this works:

> The elimination rule for '&' applies to a sentence only when an '&' occurs as the sentence's main connective. The same thing goes for 'v', '⊃', and '≡'. The components used with the main connective are the components to which the elimination rule makes reference.
>
> The elimination rule for '~' applies only to a doubly negated sentence, ~~**X;** that is, only when '~' is the sentence's main connective, and the '~' is applied to a component, ~**X,** which itself has a '~' as its main connective.
>
> The introduction rule for '&' licenses you to write as a conclusion a sentence, the main connective of which is '&'. The same thing goes for 'v', '⊃', '≡', and '~'.

EXERCISES

6–2. Give derivations which establish the validity of the following arguments:

a) (AvB)&[(AvB)⊃C]
———
C

b) A
(AvB)≡[(A⊃K)&(B⊃K)]
———
K

c) [A⊃(Dv~B)]&{[A⊃(Dv~B)]⊃(B⊃A)}
———
B⊃A

6–3. DERIVATIONS: OVERVIEW, DEFINITIONS, AND POINTS TO WATCH OUT FOR

This chapter and chapter 5 have described, explained, and illustrated derivations. Let's pull these thoughts together with some explicit definitions and further terminology.

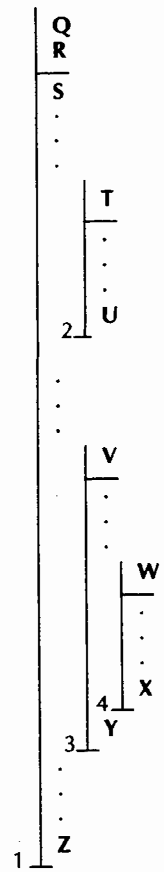
> A *Rule of Inference* tells when you are allowed, or *Licensed*, to draw a conclusion from one or more sentences or from a whole argument (as represented by a subderivation).
>
> A *Derivation* is a list of which each member is either a sentence or another derivation. If a first derivation has a second derivation as one of the first derivation's parts, the second derivation is called a *Subderivation* of the first and the first is called the *Outer Derivation* of the second. Each sentence in a derivation is a premise or assumption, or a reiteration of a previous sentence from the same derivation or an outer derivation, or a sentence which follows by one of the rules of inference from previous sentences or subderivations of the derivation.

In practice, we always list the premises or assumptions of a derivation at its beginning, and use a horizontal line to separate them from the further sentences which follow as conclusions. What's the difference between premises and assumptions? From a formal point of view, there is no difference at all, in the sense that the rules of inference treat premises and assumptions in **exactly** the same way. In practice, when an unargued sentence is assumed at the beginning of the outermost deduction, we call it a premise. When an unargued sentence is assumed at the beginning of a subderivation, we call it an assumption. The point is that we always terminate subderivations before the end of a main derivation, and when we terminate a subderivation, in some sense we have gotten rid of, or *Discharged*, the subderivation's assumptions.

To make these ideas clearer and more precise, we have to think through what the vertical lines, or *Scope Lines*, are doing for us?

> A *Scope Line* tells us what sentences and subderivations hang together as a single derivation. Given a vertical scope line, the derivation it marks begins where the line begins and ends where the line ends. The derivation marked by a scope line includes all and only the sentences and subderivations immediately to the right of the scope line.

To help sort out these definitions, here is a schematic example:



Notice that at the bottom of each scope line I have written a number to help us in talking about the different component derivations. Consider first the main derivation, derivation 1, marked with the leftmost scope line numbered '1' at its bottom. Derivation 1 includes premises **Q** and **R** and has a first conclusion **S,** oher conclusions not explicitly shown, indicated by . . . , and the final conclusion **Z.** Derivation 1 also includes two subderivations, derivations 2 and 3. Derivation 2 has assumption **T,** various conclusions not explicitly indicated (again signaled by . . .), and final conclusion **U.** Derivation 3 starts with assumption **V,** has final conclusion **Y,** and includes a subderivation of its own, derivation 4.

This organization serves the purpose of keeping track of what follows from what. In the outermost derivation 1, all the conclusions of the derivation (**S** . . . **Z**) follow from the derivation's premises, **Q** and **R**. This means that every assignment of truth values to sentence letters which makes the premises **Q** and **R** true will make all the conclusions of derivation 1 true. But the conclusions of a subderivation hold only under the subderivation's additional assumption. For example, the conclusion **U** of subderivation 2 is subject to the assumption **T** as well as the premises **Q** and **R**. This means that we are only guaranteed that any assignment of truth values to sentence letters which makes **Q, R, and T** all true will make **U** true also. In other words, when we start a subderivation, we add an additional assumption which is assumed in effect just in the subderivation. Any premises or assumptions from outer derivations also apply in the subderivation, since they and their consequences can be reiterated into the subderivation.

You should particularly notice that when a subderivation has ended, its special assumption is no longer assumed. It is not assumed in any conclusions drawn as part of the outer derivation, nor is it assumed as part of a new subderivation which starts with a different assumption. Thus the truth of **T** is not being assumed anywhere in derivation 1, 3, or 4. This is what we mean by saying that the assumption of a subderivation has been discharged when the subderivation is terminated.

These facts are encoded in the reiteration rule which we can now spell out more clearly than before. The reiteration rule spells out the fact that a subderivation assumes the truth, not only of its own assumption, but of the prior assumptions, premises, and conclusions of any outer derivation. Thus, in subderivation 2, reiteration permits us to write, as part of 2, **Q, R, S,** or any other conclusion of 1 which appears before 2 starts. This is because inside 2, we assume that the premises of outer derivation 1 are true. And because whenever the premises are true, conclusions which follow from them are true, we may also use the truth of any such conclusions which have already followed from these premises.

But we cannot reiterate a sentence of 2 in, for example, 1. This is because when we end subderivation 2 we have discharged its premise. That is, we are no longer arguing under the assumption that the assumption of 2 is true. So, for example, it would be a mistake to reiterate **U** as part of 1. **U** has been proved only subject to the additional assumption **T**. In 1, **T** is not being assumed. In the same way, we cannot reiterate **U** as part of 3 or 4. When we get to 3, subderivation 2 has been ended. Its special assumption, **T,** has been discharged, which is to say that we no longer are arguing under the assumption of **T**.

Students very commonly make the mistake of copying a conclusion of a subderivation, such as **U,** as a conclusion of an outer derivation—in our schematic example, listing **U** as a conclusion in derivation 1 as well as in subderivation 2. I'll call this mistake the mistake of hopping scope lines. **Don't hop scope lines!**

We can, however, reiterate **Q, R, S,** or any prior conclusion in 1 within sub-sub-derivation 4. Why? Because 4 is operating under its special assumption, **W,** as well as all the assumptions and premises of **all** derivations which are outer to 4. Inside 4 we are operating under all the assumptions which are operative in 3, which include not only the assumption of 3 but all the premises of the derivation of which 3 is a part, namely, 1. All this can be expressed formally with the reiteration rule, as follows: To get a premise or prior conclusion of 1 into 4, first reiterate the sentence in question as part of 3. Now that sentence, officially part of 3, can be reiterated again in 4. But we can dispense with the intermediate step.

Incidentally, once you clearly understand the reiteration rule, you may find it very tiresome always to have to explicitly copy the reiterated sentences you need in subderivations. Why, you may wonder, should you not be allowed, when you apply other rules, simply to appeal to prior sentences in outer derivations, that is, to the sentences which the reiteration rule allows you to reiterate? If you fully understand the reiteration rule, you will do no harm in thus streamlining your derivations. I will not use this abbreviation, because I want to be sure that all of my readers understand as clearly as possible how reiteration works. You also should not abbreviate your derivations in this way unless your instructor gives you explicit permission to do so.

Scope lines also indicate the sentences to which we can apply a rule in deriving a conclusion in a derivation or subderivation. Let us first focus on rules which apply only to sentences, that is, rules such as ∨E or ⊃E, which have nothing to do with subderivations. The crucial feature of such a rule is that, if the sentences to which we apply it are true, the conclusion will be true also. Suppose, now, we apply such a rule to the premises **Q** and **R** of derivation 1. Then, if the premises are true, so will the rule's conclusion, so that we can write any such conclusion as part of derivation 1. In further application of such rules in reaching conclusions of derivation 1, we may appeal to 1's prior conclusions as well as its premises, since if the premises are true, so will the prior conclusions. In this way we are still guaranteed that if the premises are true, so will the new conclusion.

But we **can't** apply such a rule to assumptions or conclusions of a subderivation in drawing conclusions to be made part of derivation 1. For example, we can't apply a rule to sentences **S** and **U** in drawing a conclusion which will be entered as a part of derivation 1. Why not? Because we want all the conclusions of 1 to be guaranteed to be true if 1's premises are true. But assumptions or conclusions of a subderivation, say, 2, are only sure to be true if 1's premises **and** 2's special assumption are true.

In sum, when applying a rule of inference which provides a conclusion when applied to sentences ("input sentences"), the input sentences must already appear before the rule is applied, and all input sentences as well as the conclusion must appear in the **same** derivation. Violating this instruction constitutes a second form of the mistake of hopping scope lines.

What about ⊃I and ~I, which don't have sentences as input? Both these rules have the form: If a subderivation of such and such a form appears in a derivation, you may conclude so and so. It is important to appreciate that these two rules **do not** appeal to the sentences which appear in the subderivation. They appeal to the subderivation as a whole. They appeal not to any particular sentences, but to the fact that from one sentence we have derived certain other sentences. That is why when we annotate these rules we cite the whole subderivation to which the rule applies, by indicating the inclusive line numbers of the subderivation.

Consider ⊃I. Suppose that from **T** we have derived **U**, perhaps using the premises and prior conclusions of our outer derivation. Given this fact, any assignment of truth values to sentence letters which makes the outer derivation's premises true will also make the conditional **T⊃U** true. (I explained why in the last chapter.) Thus, given a subderivation like 2 from **T** to **U**, we can write the conclusion **T⊃U** as part of the outer derivation 1. If 1's premises are true, **T⊃U** will surely be true also.

The key point to remember here is that when ⊃I and ~I apply to a subderivation, the conclusion licensed appears in the same derivation in which the input subderivation appeared as a part. Subderivation 2 licenses the conclusion **T⊃Y** as a conclusion of 1, by ⊃I; and ⊃I, similarly applied to derivation 4, licenses concluding **W⊃X** as part of 3, but **not** as part of 1.

By this time you may be feeling buried under a pile of details and mistakes to watch out for. Natural deduction may not yet seem all that natural. But, as you practice, you will find that the bits come to hang together in a very natural way. With experience, all these details will become second nature so that you can focus on the challenging task of devising a good way of squeezing a given conclusion out of the premises you are allowed to use.

EXERCISES

6–3. For each of the following arguments, provide a derivation which shows the argument to be valid. If you get stuck on one problem, try another. If you get good and stuck, read over the examples in this chapter, and then try again.

a) R
———
(RvD)&(RvK)

b) (~A&B)vC
AvD
———
~C⊃D

c) ~(Hv~D)
F⊃H
———
~F

d) F⊃(O⊃M)
———
(F&O)⊃M

e) P&~Q
———
~R⊃~[P⊃(RvQ)]

f) (K&G)⊃S
———
K⊃(G⊃S)

g) ~(A&~F)
D⊃A
———
D⊃F

h) ~(N⊃I)
~I⊃C
———
C

i) ~M⊃~L
~L⊃~K
———
K⊃M

j) QvF
Q⊃A
F⊃A
———
A

k) ~(S&T)
SvT
———
~S≡T

l) ~C⊃(AvB)
~D⊃(Cv~B)
~(CvD)
———
(AvB)&(Cv~B)

m) G≡~H
———
~G≡H

n) P≡Q
———
~(P≡~Q)

o) (N⊃S)&(G⊃D)
(SvD)⊃{[F⊃(FvK)]⊃(N&G)}
———
N≡G

p) (S&B)⊃K
(G⊃P)&(GvP)
~B≡(~P&G)
———
S⊃K

q) CvB
~(C&~B)
~(~C&B)
———
C&(Bv~C)

r) H⊃(D⊃K)
(K&M)⊃P
I⊃~(M⊃P)
———
H⊃(D⊃~I)

6–4. Write a rule of inference for the Sheffer stroke, defined in section 3–5.

CHAPTER SUMMARY EXERCISES

This chapter has focused on improving your understanding of material introduced in chapter 5, so there are only a few new ideas. Complete short explanations in your notebook for the following terms. But also go back to your explanations for the terms in the chapter summary exercises for chapter 5 and see if you can now explain any of these terms more accurately and clearly.

a) Reductio Ad Absurdum Strategy
b) Main Connective
c) Discharging an Assumption
d) Hopping Scope Lines