APPENDIX B
Relations and Orderings

We begin this appendix with a very brief review of relations in general. The concept of the **ordered pair** (x, y) of any two objects x and y may be left as an undefined concept, or it may be defined rigorously, as in Section 2.3. Once we have ordered pairs at our disposal, we can iterate the process to define ordered triples, and ordered n-tuples in general. Specifically, (x, y, z) is usually defined to be $((x, y), z)$, although $(x, (y, z))$ would work just as well.

We can then define the **Cartesian product** of two sets A and B by $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$. We also have extended Cartesian products; for example, $A \times B \times C$ means $(A \times B) \times C$ or, equivalently, $\{(x, y, z) \mid x \in A \text{ and } y \in B \text{ and } z \in C\}$. One writes A^2 for $A \times A$, A^3 for $A \times A \times A$, etc. This notation is technically ambiguous, since A^n also means the set of all functions from n to A. But in many situations, the difference between the two possible sets denoted A^n is not significant.

An n-ary **relation** is simply a set of ordered n-tuples. The words "unary," "binary," and "ternary" mean 1-ary, 2-ary, and 3-ary, respectively. Note that a unary relation is just a set. Without any adjective, the word "relation" usually means a binary relation. An n-ary relation on a set A is defined to be any subset of A^n.

It is common in mathematics to use the word "relation" for a statement with free variables that is used to define a set of ordered n-tuples. For instance, we might say "Consider the relation $x < y$ on the set of real numbers," or simply "Consider the less-than relation on $\mathbb{R}$." Technically this refers to the relation $\{(x, y) \mid x, y \in \mathbb{R} \text{ and } x < y\}$. It is more precise to call $x < y$ a binary **predicate** that we are using to define a relation, but it is often not important to worry about this distinction.

Orderings

For the rest of this appendix, we assume that R is a binary relation. We write xRy as an abbreviation for $(x, y) \in R$. As usual, the **domain** of R is $Dom(R) = \{x \mid \exists y(xRy)\}$ and the **range** of R is $Rng(R) = \{y \mid \exists x(xRy)\}$.

Definitions. We say that R is:

reflexive (on A) if xRx, for all $x \in A$;

antisymmetric if, whenever xRy and yRx, then $x = y$;

transitive if, whenever xRy and yRz, then xRz;

a **preordering** (on A) if it is reflexive and transitive;

a **partial ordering** (on A) if it is an antisymmetric preordering;

a **total ordering** or a **linear ordering** or a **chain** (on A) if it is a partial ordering and also satisfies **trichotomy**: for any $x, y \in A$, either xRy, yRx, or $x = y$. This last condition may also be described by saying that any two elements of A are **comparable** under R.

The words "on A" are in parentheses in several of these definitions because they are often omitted in practice. When that occurs, the usual implication is that the unmentioned A is $Dom(R) \cup Rng(R)$.

If R is a partial ordering, it is common to write $x \leq y$ for xRy. We can then write $x \geq y$ for $y \leq x$, $x < y$ for $x \leq y \wedge x \neq y$, and $x > y$ for $y < x$. The relation $\geq$ defines a new partial ordering on A; it is simply R^{-1}. (Of course, it's permissible to reverse all of this by writing $x \geq y$ for xRy and $x \leq y$ for the inverse relation.)

On the other hand, the relations defined by $<$ and $>$ are not partial orderings as defined above. Rather, they are **irreflexive** partial orderings, meaning that they are transitive and **irreflexive**: $x < x$ is always false. From this follows **strong antisymmetry**: whenever $x < y$ holds, then $y \nless x$. Conversely, if S is an irreflexive partial ordering on A, then the relation on A defined by (xSy or $x = y$) is a (reflexive) partial ordering on A. Furthermore, a reflexive partial ordering is total if and only if the associated irreflexive ordering is total. In other words, it is easy to go "back and forth" between reflexive and irreflexive orderings, and we will freely do so.

Definitions. Let R be a partial ordering and $x \in B \subseteq Dom(R)$. Then x is called:

a **minimal** element of B if $\sim \exists y \in B(y < x)$;
a **maximal** element of B if $\sim \exists y \in B(y > x)$;
the **least** element of B if $x \leq y$, for all $y \in B$;
the **greatest** element of B if $x \geq y$, for all $y \in B$.

The following facts are elementary: if a subset of the domain of a partially ordered set has a least element or a greatest element, then that element is unique. A least element must be a minimal element, and a greatest element must be a maximal element; in a total ordering, the converses also hold.

Definitions. A partial ordering is called **well-founded** if every nonempty subset of its domain has a minimal element. (This definition can actually be made for binary relations in general.) A well-founded total ordering is called a **well-ordering**.

So a well-ordering is a total ordering in which every nonempty subset of the domain has a least element.

Definitions. If A is a partially ordered set, $B \subseteq A$, and $x \in A$, we say that x is an **upper bound** of B if $y \leq x$ for every $y \in B$. If B has a least upper bound, then it is of course unique and is denoted LUB(B) or Sup(B) (the **supremum** of B). In a totally ordered set, a subset with no upper bound is said to be **unbounded above** or **cofinal**.

Similarly, we define what is meant by a **lower bound** of B. If B has a greatest lower bound, then it is unique and is denoted GLB(B) or

$Inf(B)$ (the **infimum** of B). In a totally ordered set, a subset with no lower bound is said to be **unbounded below** or **coinitial**.

Notation. In any total ordering, we can define the usual types of **bounded intervals** (a, b), $[a, b]$, $[a, b)$, and $(a, b]$. We can also define **rays**: the type of sets that, in $\mathbb{R}$, would be denoted $(-\infty, b)$, $[a, \infty)$, etc.

We will use the term **interval** to mean either a bounded interval or a ray. Also, the **initial segment** defined by an element a means the ray $\{x \mid x < a\}$. This term is usually applied only to well-orderings.

Definitions. Let a be an element of a totally ordered set. The **immediate successor** of a is the least element that is greater than a. The **immediate predecessor** of a is defined similarly.

The immediate successor and predecessor of an element are obviously unique, if they exist. Also, b is the immediate successor of a if and only if a is the immediate predecessor of b.

Definition. A total ordering is called **discrete** if every element that is not the greatest (respectively, least) element in the ordering has an immediate successor (respectively, predecessor).

Definition. Suppose that A is a totally ordered set and $B \subseteq A$. We say that B is a **dense** subset of A if, whenever $x, y \in A$ and $x < y$, there exists $z \in B$ such that $x < z < y$.

A total ordering is called dense if its domain has at least two members and is a dense subset of itself. The second conjunct simply means that no element has an immediate successor or an immediate predecessor.

Every subset of the domain of a discrete ordering is again discrete under the restriction of the original ordering. Every *interval* with more than one element (but not every subset) in a dense ordering is again dense under the restriction of the original ordering.

Discrete orderings and dense orderings may be thought of as opposite ends of a spectrum. No ordering is both discrete and dense.

Example 1. Every total ordering on a finite set is discrete. The usual ordering on $\mathbb{Z}$ is discrete; therefore, by the previous remark, so is the usual ordering on $\mathbb{N}$.

$\mathbb{Q}$ is a dense subset of $\mathbb{R}$. It follows that the usual orderings on $\mathbb{R}$ and $\mathbb{Q}$ are dense. Therefore, so are the orderings on all intervals in $\mathbb{R}$ and $\mathbb{Q}$ (except for intervals that are empty or include only one point).

There are several useful statements involving orderings that are equivalent (in ZF) to the axiom of choice. Here are the definitions of two of the most important ones:

Definition. **Zorn's lemma** states that a partial ordering in which every chain (totally ordered subset) has an upper bound must have a maximal element.

The second proof of Theorem 2.20 given in Section 2.5 illustrates the typical use of Zorn's lemma.

Definition. The **Hausdorff maximal principle** states that every chain in a partial ordering must be contained in some maximal chain.

Functions and equivalence relations

Orderings are one of the three most important types of binary relation used in mathematics. For the sake of completeness, here are the definitions of the other two.

Functions are also a type of relation. Specifically, as a set of ordered pairs, a **function** from A to B is simply a subset of $A \times B$ in which each element of A occurs in exactly one of the ordered pairs. This is the standard set-theoretic definition of a function. However, as noted at the beginning of Section 3.2, there are situations in which this definition is not quite satisfactory.

If f is a function, $f(x) = y$ is the usual way of writing $(x, y) \in f$. This is very handy because it makes it possible to write $f(x)$ as a term, and to substitute other terms for the variable x. Also, the notation $f : A \to B$ means that f is a function from A to B. Note that this implies that A is precisely the domain of f, but B can be any set that contains the range of f. Part of the reason for this convention is convenience. For instance, if f is the real-valued function defined by $f(x) = x^2 + \sin(x)$, we can write $f : \mathbb{R} \to \mathbb{R}$ without needing to take the trouble to determine the exact range of f.

The other very basic type of relation is equivalence relations. A binary relation R is said to be **symmetric** if $xRy \leftrightarrow yRx$ holds for all x and y. An **equivalence relation on** A is a relation with domain A that is reflexive, symmetric, and transitive. An equivalence relation normally expresses some way in which two objects are similar or alike. For example, the predicate "x and y were born in the same year" defines an equivalence relation on any set of people. Congruence and similarity define equivalence relations on any set of triangles or any other set of geometric shapes. The property of having the same integer part or the same decimal part defines an equivalence relation on $\mathbb{R}^+$.

If R is an equivalence relation on A and $x \in A$, the **equivalence class** of x, denoted $[x]_R$ or simply $[x]$ when there is no possibility of confusion, is the set $\{y \mid xRy\}$. Intuitively, $[x]_R$ is the set or "club" of objects that are similar to x, under R. For example, if R is the equivalence relation on people based on year of birth and Lucian was born in 1988, then $[\text{Lucian}]_R$ is the set of all people born in 1988. The most important mathematical fact about equivalence relations is that the equivalence classes must partition the domain. That is, the union of all the equivalence classes is the whole domain, and any two equivalence classes are either identical or disjoint. There can be no "partial overlap." Furthermore, this situation actually provides a one-to-one correspondence between the collection of all equivalence relations on any given set and the collection of partitions on that set.