8.5.1 Theories and models

As we said in Chapter 5, the distinction between syntax and semantics in the logical tradition is closely tied to the distinction between formal systems and their interpretations. Model theory, the study of the interpretations of formal systems, focuses on the relation between *theories* and *models*, with these terms understood in a technical sense which we will now describe.

A set of axioms together with all the theorems derivable from them is called a *theory*. Or equivalently, a theory is a set of statements that is closed under logical consequence, i.e. is such that any logical consequence of any statement in the set is again in the set.

Finding a *model* for a theory requires finding some abstract or concrete structured domain and an interpretation for all of the primitive expressions of the theory in that domain such that on that interpretation, all of the statements in the theory come out *true* for that model on that interpretation. If a theory has an axiomatic characterization, something is a model for that theory iff it is a model for the axioms.

Plane geometry is the standard model of the Euclidean axioms; before the discovery of the non-Euclidean geometries discussed in 5.1 it was believed to be the only model. The natural numbers are the standard or intended model of the Peano axioms; we will see some non-standard models in section 8.5.7 below.

In exploring theories and models, one can start at either end, and mathematical discoveries and advances have been made in both directions. One can start with a given set of phenomena as intended models and try to write down axioms that will best characterize them - this often forces one to sharpen up one's conception of the intended coverage of the theory, and of course helps to uncover various consequences of one's initial assumptions (One can take the whole enterprise of linguistics as trying to formally characterize the class of possible human languages; the starting point is then a somewhat vaguely specified set of intended models.) One can also start from a set of axioms and see what sorts of models it has. In the model-theoretic perspective, these two complementary activities constantly feed each other. Different axiomatic systems may be discovered to characterize exactly the same set of models, and hence to be equivalent from a semantic point of view; or two quite disparate domains may be discovered to have virtually identical axiomatizations, revealing a hitherto unsuspected structural similarity

A note of warning: the term "model", especially in the verbal form "modelling", has another very different sense as well, one in which it actually comes closer to what logicians mean by *theory* than to what they mean by *model*, and outside of logic and model theory this other sense may in fact be more common. When one speaks of modelling some physical phenomenon, or constructing an abstract model of some biological or cognitive process, the intent is generally some form of theory building or at least some step in that direction. One important clue to help resolve the ambiguity comes from looking at what the model in question is a model *of*: models in the sense of model theory are always models of axioms or other expressions in some language, never of concrete objects. Models themselves in model theory may be either concrete or abstract objects, so the nature of the things modelled is a more reliable clue to the relevant sense of "model" than is the nature of the models.

In the remaining subsections of this section, we will first look at some fundamentally important properties that relate formal systems and theories to their models, and then look into some examples of axiomatic systems and models for them, some very simple and some quite rich, illustrating the interplay between axioms and models as we go. In these sections we take the logic as a given; in 8.6 we will broach the issue of axiomatizing the logic itself.