8.5.5 Axioms for ordering relations

Various kinds of orderings were defined in Chapter 3 in terms of such properties of relations as transitivity, antisymmetry, etc. These definitions can be very easily formalized as axiomatic systems, with each relevant property specified by an axiom. What we gave as "examples" of the different kinds of orderings in 3.5 we can now redescribe as *models* of the corresponding axiom systems.

Any ordering relation is a binary relation R on a set S. We assume as part of the "background theory" ordinary set theory, including the representation of binary relations on S as sets of ordered pairs of members of S, and we specify the particular axioms that must be satisfied by particular kinds of orderings.

DEFINITION 8.10 *R is a weak partial order on S iff:*

1. Transitivity: $\forall x \forall y \forall z((x \in S \,\&\, y \in S \,\&\, z \in S) \rightarrow ((Rxy \,\&\, Ryz) \rightarrow Rxz))$
2. Reflexivity: $\forall x(x \in S \rightarrow Rxx)$
3. Antisymmetry: $\forall x \forall y((x \in S \,\&\, y \in S) \rightarrow ((Rxy \,\&\, Ryx) \rightarrow x = y))$

■

Alternatively, we need not explicitly assume set theory or use the language of set membership, but can simply take the domain S as the universe over which the quantified variables in the axioms range. In that case, the previous definition would be recast as follows:

DEFINITION 8.11 *R is a weak partial order on S iff:*

1. Transitivity: $\forall x \forall y \forall z((Rxy \,\&\, Ryz) \rightarrow Rxz)$
2. Reflexivity: $\forall x(Rxx)$
3. Antisymmetry: $\forall x \forall y((Rxy \,\&\, Ryx) \rightarrow x = y)$

■

One will also encounter axiomatizations in which the wide-scope universal quantifiers are omitted and open formulas are understood as universally quantified. We will not take that further step here; but it is worth noting that the prevalence of "pure universal" axioms like those above is not simply an accident. The study of model theory has shown that pure universal theories, all of whose axioms are pure universal ones like those above, have a number of nice relations to their models.

In Chapter 3 it was noted that generally each weak ordering, obeying the axioms of reflexivity and antisymmetry, could be paired with a corresponding strong ordering, with those axioms replaced by irreflexivity and asymmetry.

DEFINITION 8.12 R *is* a strict partial order on S *iff*:

1. Transitivity: $\forall x \forall y \forall z((Rxy \,\&\, Ryz) \rightarrow Rxz)$
2. Irreflexivity: $\forall x(\sim Rxx)$
3. Asymmetry: $\forall x \forall y(Rxy \rightarrow \sim Ryx)$

■

The relations R_1, R_2, and R_3 diagrammed in Section 3.5 are all models of the axioms for weak partial orders, and S_1, S_2, and S_3 are models of the axioms for strict partial orders. Another model of weak partial orders is the subset relation on any collection of sets; the 'proper subset' relation provides the corresponding strict order.

What about the relations 'is at least as old as' and 'is older than' on H, the set of humans, assuming there do exist various pairs of people who are the same age? Intuitively, one might suppose that 'is at least as old as' would be a weak partial order on humans, much as 'is a subset of' provides a weak partial order on a set of sets. But while 'is at least as old as' on the set of humans does satisfy the axioms of transitivity and reflexivity, it fails antisymmetry. For let a and b be two individuals of the same age: then Rab and Rba, but $a \neq b$.

Note carefully the role of identity here: $a \neq b$ because a and b are two distinct members of the set H; being the same age makes them equivalent with respect to the relation R (and 'is the same age as' is an equivalence relation), but it doesn't make them equal in the sense required by the antisymmetry condition

A relation like 'is at least as old as' which satisfies transitivity and reflexivity but possibly fails antisymmetry is called a *preorder* or sometimes a *quasi-order*; we could axiomatize it by writing down just the first two of the three axioms for a weak partial order. Where there's a preorder on S there is always the possibility of defining an order on a suitable partitioning of S In this example, for instance, intuitively we want to count people of the same age as identical or indistinguishable; the formal technique for achieving that is to define the ordering not directly on the set of all people but on the set of equivalence clases formed under the relation 'is the same age as', in which all the people of a given age will be grouped together in a single equivalence class. In fact, when we step back and look at these equivalence classes, we can see that one might even consider analyzing our talk of ordering people by their ages in terms of ordering people's ages.

What about 'is older than'? Does that similarly fail to be a strict partial order on the given set of humans? Actually, no; it does satisfy all three axioms of Transitivity, Irreflexivity, and Asymmetry But unlike the corresponding order on *ages*, or the apparently similar relation 'is greater than' on the numbers, it is not a linear order, since it is not connected;[2] see the following definitions.

Note: in examples such as those we have just been discussing, it is not so important to try to learn to remember the names and definitions of particular kinds of orderings or which examples satisfy which axioms; you can always look up the technical details in this or other books when you need them, and details of terminology are not all uniform among different research communities anyway The important thing to focus on in this chapter are the illustrations of how changes in the axioms relate to changes in the models, and how the interesting properties of a formal system can be explored from both syntactic and semantic perspectives, often most fruitfully by looking at both together.

Linear orderings, both weak and strict, were defined and illustrated in Section 3.5. If we recast them axiomatically, they come out as follows.

DEFINITION 8.13 *R is* a weak linear (or total) order on *S iff:*

1. Transitivity: $\forall x \forall y \forall z((Rxy \,\&\, Ryz) \rightarrow Rxz)$
2. Reflexivity: $\forall x(Rxx)$
3. Antisymmetry: $\forall x \forall y((Rxy \,\&\, Ryx) \rightarrow x = y)$
4. Connectedness: $\forall x \forall y(x \neq y \rightarrow (Rxy \vee Ryx))$

■

Given that the first three axioms above constitute the definition of a weak partial order on S, we can abbreviate the definition above as follows.

[2]The relation 'is at least as old as' *is* connected, but neither asymmetric nor antisymmetric. It is an example of what Suppes (1957) defines as a *weak ordering*, a relation which is transitive, reflexive, and connected, i.e a connected preorder. This is not a kind of ordering that is standardly singled out; but one is free to define and name whatever kinds of formal systems one thinks will prove useful for one's purposes.

DEFINITION 8.14 *R is* a weak linear (or total) order on S *iff*:

1. R *is a weak partial order* on S
2. Connectedness: $\forall x \forall y(x \neq y \rightarrow (Rxy \vee Ryx))$

■

We give the definition of strict linear order in analogous fashion.

DEFINITION 8.15 *R is a* strict linear (or total) order on S *iff*:

1. R *is a strict partial order* on S
2. Connectedness: $\forall x \forall y(x \neq y \rightarrow (Rxy \vee Ryx))$

■

Among the models for these axioms systems, the relations R_3 and S_3 given in Section 3.5 are models of weak and strict linear orderings respectively.

The reader may have noticed a certain degree of systematicity in the relation between the names chosen for various kinds of ordering relations and the selection of axioms used in their definitions. Such systematicity is most prevalent (and most desirable) in contexts where the emphasis is on contrasts among closely related axiomatic systems, as is the case here. Shorter names are often used when differences among similar systems are not at issue; so, for instance, an author may omit the adjectives *weak* and *strict* and talk simply of partial and total orderings if all her orderings are weak or if all of them are strict; definitions usually accompany initial uses of such terms when there could be any doubt. In the case of orderings, watch out for the use of the adjective *strong*, which is used as an antonym sometimes of *weak* and sometimes of *partial*. The lack of perfect standardization in nomenclature is a perfectly reasonable side effect of the useful versatility of axiomatic definitions; be prepared when in doubt to check a given author's definitions.

The definition of *well-ordering* was also given in Section 3.5: a set S is *well-ordered* by a relation R if R is a total order and, further, every subset of S has a least element in the ordering relation. If we try to write down this further condition as an additional axiom to add to the axioms for total orderings, we come across an important difference between it and all the other axioms we have introduced in this section: it cannot be expressed in first-order predicate logic

If we give ourselves the full expressive power of set theory, including the possibility of quantifying over sets, we can write down the axioms for well-ordering in the same form we used for the first version above of the definition of weak partial orderings.

DEFINITION 8 16 *A relation R is a well-ordering of a set S iff:*

1. *R is a total ordering on S.*
2. *Every subset of S has a least element with respect to the order R:* $\forall S'((S' \subseteq S) \rightarrow \exists x(x \in S' \,\&\, \forall y((y \in S' \,\&\, x \neq y) \rightarrow Rxy)))$

■

But we cannot omit the set theory talk this time as we could before. We can recast it so that we are quantifying over one-place predicates instead of over sets, which we do in Section 8.6.7 where we discuss higher-order logics. But what we cannot do is express the second axiom just with ordinary individual variables ranging over the members of the domain S

The well-ordering axiom, axiom 2 above, turns out to be quite powerful and subtle. If logicians could have found a way to replace it with a first-order axiom having the same effect, they surely would have. What has been proved is that the well-ordering axiom is equivalent to each of several other non-first-order axioms, including Peano's fifth axiom, the induction axiom, which has already been introduced and to which we will return in Section 8.5.7. The relations among these higher-order axioms are discussed in Section 8.6.7. Properties which like transitivity and reflexivity *can* be expressed by first-order axioms are called first-order properties, but the modifier is used only when the contrast with higher-order properties is relevant

Ordering relations and their axiomatic characterizations provide a rich round for exploring the syntactic and the semantic side of formal systems and their interrelations. Once one sees that each property like reflexivity or antisymmetry can be characterized by an axiom, the possible combinations to be explored become endless. Can an ordering be both asymmetric and antisymmetric? Does the answer to that question vary with the other axioms in the given system? Are there axioms that will force the set ordered to be infinite? To be finite? Are there informally describable kinds of orderings that cannot be characterized by a finite set of axioms? Is that last question well-defined, and if it is not precise, can it still be fruitful?

The rich realm of axiomatizations of ordering relations also leads one to wonder whether there is some single most general characterization of orderings such that all the well-known kinds of orderings are gotten by adding various axioms to some common core of shared axioms. Different authors have different degrees of tolerance on this question; the natural desire for a most general notion of ordering is in conflict with the fact that the standard kinds of ordering relations are required to be, besides transitive, either reflexive and antisymmetric or irreflexive and asymmetric and there seems to be no non-ugly way to say just that. Suppes (1957), noting that transitivity is the one property they all share, makes transitive relations the most general case in a diagram displaying the inclusion relation among several different kinds of ordering relations (an ordering of ordering relations.) Most authors decline to attempt a single most general definition of ordering relations. A wealth of syntactic and semantic arguments establishing various properties of orderings can be found in Suppes (1960).