7.6 Formal and informal proofs

We may apply the principles developed in the preceding section on Natural Deduction to the proof of statements about sets. Note that $A \subseteq B$, for example, is a statement which asserts that a certain two-place predicate, 'is a subset of', holds of a particular pair of sets, A and B. That this is customarily written $A \subseteq B$ instead of $\subseteq (A, B)$ is merely a notational convention of set theory. Similarly, $x \in A$ is an open statement containing the variable x in which "$\in A$" functions as a one-place predicate $(\exists x)(x \in A)$ is then a statement asserting that A is not empty The Axiom of Extension (two sets are equal if they have the same members) might be written as $(\forall X, Y)(X = Y \leftrightarrow (\forall x)(x \in X \leftrightarrow x \in Y))$.

The following is a proof showing formally that $(\forall X, Y)(X = Y \leftrightarrow (X \subseteq Y \,\&\, Y \subseteq X))$ (two sets are equal iff each is a subset of the other) follows from the Axiom of Extension as premise:

(7–52)
1. $(\forall X, Y)(X = Y \leftrightarrow (\forall x)(x \in X \leftrightarrow x \in Y))$
2. $V_1 = V_2 \leftrightarrow (\forall x)(x \in V_1 \leftrightarrow x \in V_2)$ 1, U.I. (twice)
3. $V_1 = V_2 \leftrightarrow (\forall x)((x \in V_1 \rightarrow x \in V_2) \,\&\, (x \in V_2 \rightarrow x \in V_1))$ 2, Bicond.
4. $V_1 = V_2 \leftrightarrow ((\forall x)(x \in V_1 \rightarrow x \in V_2) \,\&\, (\forall x)(x \in V_2 \rightarrow x \in V_1))$ 3, Quant. Distr. (Law 4)
5. $V_1 = V_2 \leftrightarrow (V_1 \subseteq V_2 \,\&\, V_2 \subseteq V_1)$ 4, Definition of $\subseteq$

In step 5 we have simply replaced two subexpressions of line 4 by their abbreviated forms.

6. $(\forall X, Y)(X = Y \leftrightarrow (X \subseteq Y \,\&\, Y \subseteq X))$5, U.G (twice)

Line 6 thus can be added to our stock of true statements about sets in general (cf. Fig 1-7).

As another illustration of a proof of a set-theoretic proposition we demonstrate the following (which was asserted without proof in Ch. 1, Sec. 4):

> **For any sets X, Y, and Z, if X is a subset of Y and Y is a subset of Z, then X is a subset of Z.**

In symbols,

(7–53) $(\forall X, Y, Z)((X \subseteq Y \,\&\, Y \subseteq Z) \rightarrow X \subseteq Z)$

Our demonstration uses a conditional proof:

(7–54)	1.	$V_1 \subseteq V_2 \,\&\, V_2 \subseteq V_3$	Aux Premise
	2.	$(\forall x)(x \in V_1 \rightarrow x \in V_2) \,\&\, (\forall x)(x \in V_2 \rightarrow x \in V_3)$	1, Def. of $\subseteq$
	3.	$(\forall x)((x \in V_1 \rightarrow x \in V_2) \,\&\, (x \in V_2 \rightarrow x \in V_3))$	2, Quant. Distr. (Law 2)
	4.	$(v \in V_1 \rightarrow v \in V_2) \,\&\, (v \in V_2 \rightarrow v \in V_3)$	3, U.I.
	5.	$v \in V_1 \rightarrow v \in V_2$	4, Simp.
	6.	$v \in V_2 \rightarrow v \in V_3$	5, Simp.
	7.	$v \in V_1 \rightarrow v \in V_3$	5, 6, H.S.
	8.	$(\forall x)(x \in V_1 \rightarrow x \in V_3)$	7, U.G.
	9.	$V_1 \subseteq V_3$	8, Def. of $\subseteq$
	10.	$(V_1 \subseteq V_2 \,\&\, V_2 \subseteq V_3) \rightarrow V_1 \subseteq V_3$	1–9, C.P.
	11.	$(\forall X, Y, Z)((X \subseteq Y \,\&\, Y \subseteq Z) \rightarrow X \subseteq Z)$	10, U.G (three times)

7.7 Informal style in mathematical proofs

Mathematicians rarely present proofs in the completely formal style we have been using since they can assume that their audience is familiar enough with logical equivalences and rules of inference to require only an outline of the essential steps. We have already used this style of presentation in earlier sections (see, for example, Chapter 3, Sec. 6). Such an informal proof should be easily expanded into a fully formal version that can be checked step by step if there is any doubt concerning its validity. Thus, the term "informal", when applied to proofs, does not mean "sloppy", only "condensed".

To illustrate, we give (7-54) as a mathematician might write it:

(7–55) Let X, Y, and Z be arbitrary sets such that $X \subseteq Y$ and $Y \subseteq Z$. Let x be an arbitrary member of X. Because $X \subseteq Y, x \in Y$; and because $Y \subseteq Z$, $x \in Z$. Therefore, $x \in X \rightarrow x \in Z$, and thus $X \subseteq Z$

Observe that no explicit mention is made of U.I and U.G., it being understood from the context and use of the word 'arbitrary' that the result is true of all sets. In the last two sentences of the proof it is assumed that the reader knows the definition of $\subseteq$ and the inference rule of Hypothetical Syllogism. The whole is in the form of a conditional proof headed by the statement $X \subseteq Y \,\&\, Y \subseteq Z$, but it is left to the reader to draw the conclusion $(X \subseteq Y \,\&\, Y \subseteq Z) \rightarrow X \subseteq Z$ and to generalize it universally.

As another example, we state the definition of 'proper subset' and give both formal and informal proofs of a theorem containing this predicate.

(7–56) $(\forall X, Y)(X \subset Y \leftrightarrow (X \subseteq Y \,\&\, X \neq Y))$

The expression $X \neq Y$ is an alternative notation for $\sim(X = Y)$. Similarly, $X \not\subseteq Y, X \not\subset Y$, and $x \notin Y$ can be written in place of $\sim(X \subseteq Y)$, $\sim(X \subset Y)$, and $\sim(x \in Y)$, respectively. The predicate $\subset$ in (7-56) is defined in terms of the predicates $\subseteq$ and $=$, which can in turn be expressed in terms of the predicate $\in$, thus:

(7–57) $(\forall X, Y)(X \subset Y \leftrightarrow ((\forall x)(x \in X \rightarrow x \in Y) \,\&$
$\sim(\forall x)(x \in X \leftrightarrow x \in Y)))$

We wish to prove:

For any sets X and Y, if X is a proper subset of Y, there is some member of Y that is not a member of X.

That is,

(7–58) $(\forall X, Y)(X \subset Y \rightarrow (\exists x)(x \in Y \,\&\, x \notin X))$

(7–59) *Proof* (formal):

1.	$V_1 \subset V_2$	Aux. Premise
2.	$V_1 \subseteq V_2 \,\&\, V_1 \neq V_2$	1, Def. of $\subset$
3.	$V_1 \neq V_2$	2, Simp.
4.	$\sim(V_1 \subseteq V_2 \,\&\, V_2 \subseteq V_1)$	3, (7-52) above
5.	$V_1 \not\subseteq V_2 \vee V_2 \not\subseteq V_1$	4, DeM.
6.	$V_1 \subseteq V_2$	2, Simp.
7.	$V_2 \not\subseteq V_1$	5, 6, D.S.
8.	$\sim(\forall x)(x \in V_2 \rightarrow x \in V_1)$	7, Def. of $\subseteq$
9.	$(\exists x) \sim(x \in V_2 \rightarrow x \in V_1)$	8, Quant. Neg.
10.	$(\exists x) \sim(x \notin V_2 \vee x \in V_1)$	9, Cond.
11.	$(\exists x)(x \in V_2 \,\&\, x \notin V_1)$	10, DeM.
12.	$V_1 \subset V_2 \rightarrow (\exists x)(x \in V_2 \,\&\, x \notin V_1)$	1–11, Cond. Proof
13.	$(\forall X, Y)(X \subset Y \rightarrow (\exists x)(x \in Y \,\&\, x \notin X))$	12, U.G. (twice)

(7–60) *Proof* (informal): Let X and Y be arbitrary sets such that $X \subset Y$. Then, by definition, $X \subseteq Y$ and $X \neq Y$. $X = Y$ iff $X \subseteq Y$ and $Y \subseteq X$. Therefore, since $X \neq Y$ and $X \subseteq Y$, it follows that $Y \not\subseteq X$, which implies that there is some x in Y that is not in X.

As a final example we give formal and informal versions of a proof involving binary relations:

For any binary relation R, $R = (R^{-1})^{-1}$.

We make use of the result proved in (7-52), i.e., for all sets X and Y, $X = Y$ iff $(X \subseteq Y \,\&\, Y \subseteq X)$. Thus we first prove $R \subseteq (R^{-1})^{-1}$, then that $(R^{-1})^{-1} \subseteq R$. (This is the customary procedure in showing equality of two sets.)

(7–61) *Proof* (formal):

1.	$\mid \langle v_1, v_2\rangle \in V$	Aux. Premise
	[$\langle v_1, v_2\rangle$ is an arbitrarily chosen ordered pair in the arbitrarily chosen binary relation V]	
2.	$\mid (\forall R)(\forall x, y)(\langle x, y\rangle \in R \leftrightarrow \langle y, x\rangle \in R^{-1})$	Def. of inverse
3.	$\mid (\forall x, y)(\langle x, y\rangle \in V \leftrightarrow \langle y, x\rangle \in V^{-1}$	2, U.I.
4.	$\mid \langle v_1, v_2\rangle \in V \leftrightarrow \langle v_2, v_1\rangle \in V^{-1}$	3, U.I. (twice)
5.	$\mid (\langle v_1, v_2\rangle \in V \rightarrow \langle v_2, v_1\rangle \in V^{-1})\,\&$ $(\langle v_2, v_1\rangle \in V^{-1} \rightarrow \langle v_1, v_2\rangle \in V)$	4, Bicond.
6.	$\mid \langle v_1, v_2\rangle \in V \rightarrow \langle v_2, v_1\rangle \in V^{-1}$	5, Simp.
7.	$\mid \langle v_2, v_1\rangle \in V^{-1}$	1, 6, M.P.
8.	$\mid (\forall x, y)(\langle x, y\rangle \in V^{-1} \leftrightarrow \langle y, x\rangle \in (V^{-1})^{-1}$	2, U.I.
	[generalizing line 2 again, this time with respect to V^{-1}]	
9.	$\mid \langle v_2, v_1\rangle \in V^{-1} \leftrightarrow \langle v_1, v_2\rangle \in (V^{-1})^{-1}$	8, U.I. (twice)
10.	$\mid (\langle v_2, v_1\rangle \in V^{-1} \rightarrow \langle v_1, v_2\rangle \in (V^{-1})^{-1})\ \&$ $(\langle v_1, v_2\rangle \in (V^{-1})^{-1} \rightarrow \langle v_2, v_1\rangle \in V^{-1})$	9, Bicond.
11.	$\mid \langle v_2, v_1\rangle \in V^{-1} \rightarrow \langle v_1, v_2\rangle \in (V^{-1})^{-1}$	10, Simp.
12.	$\mid \langle v_1, v_2\rangle \in (V^{-1})^{-1}$	7, 11, M.P.
13.	$\langle v_1, v_2\rangle \in V \rightarrow \langle v_1, v_2\rangle \in (V^{-1})^{-1}$	1-12, C.P.
14.	$(\forall x, y)(\langle x, y\rangle \in V \rightarrow \langle x, y\rangle \in (V^{-1})^{-1})$	13. U.G (twice)
15.	$V \subseteq (V^{-1})^{-1}$	14, Def. of $\subseteq$
16.	$(\forall R)R \subseteq (R^{-1})^{-1}$	15, U.G.

The proof of the other half, i.e. $(R^{-1})^{-1} \subseteq R$, is quite similar and is left as an exercise for the reader.

Here is an informal version of the part just proved:

(7–62) *Proof* (informal): Let R be an arbitrarily chosen binary relation. Assume $\langle x, y \rangle \in R$. Then by the definition of inverse, $\langle y, x \rangle \in R^{-1}$. Again, by the definition of inverse, if $\langle y, x \rangle \in R^{-1}$, then $\langle x, y \rangle \in (R^{-1})^{-1}$. Thus, if $\langle x, y \rangle \in R, \langle x, y \rangle \in (R^{-1})^{-1}$, and so $R \subseteq (R^{-1})^{-1}$.

In fact, if the proof were intended for readers assumed to be very familiar with these notions, it might appear in even more condensed form:

(7–63) *Proof*: Let R be a relation and let $\langle x, y \rangle$ be in R. Then $\langle y, x \rangle \in R^{-1}$ and $\langle x, y \rangle \in (R^{-1})^{-1}$. $\therefore R \subseteq (R^{-1})^{-1}$.

or even

(7–64) *Proof*: Obvious.

A proof is in part a demonstration that some statement follows by logical steps from assumed premises, but it is also an attempt to convince *some actual or imagined audience* of this logical connection. Therefore, what counts as an adequate proof depends to a certain extent on the level of sophistication of one's audience. Of course, as a minimal condition it must be valid, but a proof at the level of detail appropriate for an introductory logic textbook would strike an experienced mathematician as tedious and pedantic, whereas condensed proofs omitting many logical steps appear incomprehensible to beginners. In subsequent proofs in this book we will aim for an informal level which we hope will be neither condescending nor obscure.