8.6.4 About completeness proofs

This axiomatization of first-order predicate logic is complete, i.e., all and only valid formulas are provable. Although this claim itself has been proven formally by Kurt Gödel and later also by Leon Henkin, these proofs are quite technical and not directly useful for any linguistic purposes. But the notion of completeness is an important meta-theoretical concept and to get an impression of its value we discuss the main ideas of Henkin's proof semi-formally here.

The main stages of Henkin's completeness proof are the following three claims:

(1) if a formula φ is not provable in predicate logic, then the singleton set $\{\sim \varphi\}$ is consistent.

(2) every consistent set of statements M is contained in a maximally consistent set M^*.

(3) every maximally consistent set M^* has an interpretation making exactly all statements in M^* true.

We explain the three claims and the notions used in them non-technically:

ad 1) Suppose that φ is not provable in predicate logic. We reason in a *reductio ad absurdum* argument If $\{\sim\varphi\}$ is inconsistent, then, according to the definition of inconsistency, we can derive some formula ψ and its negation hence also the contradiction $(\sim\psi\,\&\,\psi)$ from it. In that case one can show that $\sim\varphi\rightarrow(\sim\psi\,\&\,\psi)$ is provable, and that therefore $\sim(\sim\psi\,\&\,\psi)\rightarrow\varphi$ is provable. Since $\sim(\sim\psi\,\&\,\psi)$ is provable as well, it follows by Modus Ponens that φ is provable after all, which contradicts the initial assumption that φ is not provable in predicate logic So the additional assumption that $\sim\varphi$ is inconsistent cannot be right, so $\sim\varphi$ is consistent.

ad 2) A set of formulas M^* is *maximally consistent* if M^* is consistent and for every arbitrary formula φ not in M^*, $M^*\cup\{\varphi\}$ is inconsistent This means that there is no formula which can be added to M^* while keeping it consistent Any consistent set of formulas M can be extended to a maximally consistent set M^* which contains all formulas of M. We enumerate the formulas $\varphi_1,\varphi_2,\varphi_3,\ldots$ in M according to their length and by equal length alphabetically and enumerate also all the individual constants. Take $M_0=M$ and form M_{n+1} for arbitrary n from the set M_n by adding the formula $\psi(a)\rightarrow(\forall x)\psi(x)$ if φ_{n+1} is of the form $(\forall x)\psi(x)$; where a is the first individual constant in the enumeration which does not occur in φ_{n+1} nor in any of the formulas in M_n. If φ_{n+1} is not of the form $(\forall x)\psi(x)$ then $M_{n+1}=M_n$. This procedure produces sequence of consistent sets $M_0,M_1,M_2,\ldots$ Let M^* be the set consisting only of all elements of any M_n. Then M_n is consistent, since there is no finite subset which is inconsistent. M^* is maximal since any formula not in M^* is excluded because it would make it inconsistent by the procedure.

ad 3) Every consistent set of formulas has a model in which all formulas are true. We should describe this precisely for any form of formula, but the details are not particularly illuminating In case the formula is universally quantified, the procedure of constructing M^* guarantees that all assignments to the quantified variable give formulas which are still in M^*.

Now if $\varphi_1,\ldots,\varphi_n\rightarrow\psi$ is valid in predicate logic, then the proof of ψ from premises $\varphi_1,\ldots,\varphi_n$ must exist in our axiomatization of predicate logic. For if $\varphi_1,\ldots,\varphi_n\rightarrow\psi$ is valid, then $\varphi_1\rightarrow(\varphi_2\rightarrow\ldots(\varphi_n\rightarrow\psi))\ldots)$ is true and hence provable. With Modus Ponens applied n times we prove ψ from $\varphi_1,\ldots,\varphi_n$.

This sketch of Henkin's completeness proof may give you a taste of one of the most important results in predicate logic. It shows that you may safely switch back and forth between model-theoretic arguments and proofs, since they are simply semantic and syntactic counterparts Perhaps the most important and startling "side effect" of the research on completeness was the discovery of negative results showing the incompleteness of some systems