8.2 Axiomatic systems and derivations

Recursive definitions and axiomatic systems have a similar logical structure. From a finite number of statements given initially an infinite number of additional statements are derivable by repeated application of a specified set of rules. The statements assumed at the outset are the *axioms*, and the additional statements, called *theorems*, are derived from the axioms and previously derived theorems by interated applications of the *rules of inference*. The set of axioms, the set of rules of inference, and the alphabet in which all these are written constitute an *axiomatic system*. Viewed in this way, a recursive definition is like an axiomatic system in which the base states the axioms and the recursion step constitutes the rules of inference. The members of the set specified by the recursive definition, aside from those given by the base, comprise the theorems of the system.

DEFINITION 8.1 *An* axiomatic system *is an ordered triple* (A, S, P) *in which*

1. A *is a finite set of symbols, called the* alphabet.
2. S *is a set of strings on* A*, called the* axioms.
3. P *is a set of* n*-place relations in* A^**, the set of all strings made from the alphabet* A*, where* $n \geq 2$ *(i.e., the* n*-tuples in* P *must be at least ordered pairs.) The members of* P *are called* productions *or* rules (of inference).

■

We now indicate how the productions are to be employed in deriving additional strings.

DEFINITION 8.2 *Given an axiomatic system* (A, S, P)*, if*

$$(x_1, x_2, \ldots, x_{n-1}, x_n)$$

is a production in P*, we say that* x_n follows from $(x_1, x_2, \ldots, x_{n-1})$. *We also use* $x_1, x_2, \ldots, x_{n-1} \rightarrow x_n$ *as an equivalent notation for* $(x_1, x_2, \ldots, x_{n-1}, x_n)$. ■

DEFINITION 8.3 *Given an axiomatic system* (A, S, P)*, a linearly ordered sequence of strings* $y_1, y_2, \ldots, y_m$ *is called a* derivation *or* proof *of* y_m *if and only if every string in the sequence is either (1) an axiom, or (2) follows one of the productions in* P *from one or more strings preceding it in the sequence. If there is a derivation of* y *in a given axiomatic system,* y *is called a* theorem *of that system.* ■

We can illustrate these definitions by reinterpreting the recursive definition in (8-1) of mirror-image strings on $\{a, b\}$ as an axiomatic system.

(8–8) $A = \{a, b\}$
$S = \{(aa, bb)\}$
$P = \{(x, y) \in A^* \times A^* \mid y = axa \vee y = bxb\}$

The productions are thus the infinite set of ordered pairs

(8–9) $\{(e, aa), (e, bb), (a, aaa), (a, bab), (b, aba), (b, bbb), (aa, aaaa), \ldots\}$

or in the alternative notation,

(8–10) $\{e \to aa, e \to bb, a \to aaa, a \to aba, b \to bbb, aa \to aaaa, \ldots\}$

In this axiomatic system, we see that the sequence of lines

(8–11) $bb, abba, aabbaa$

is a derivation of $aabbaa$ since the last string follows from preceding strings (in fact, from just the one immediately preceding) by the production $abba \to aabba$; similarly, $abba$ follows from bb by the production $bb \to abba$; and bb is an axiom Therefore, $aabbaa$ is a theorem of this axiomatic system. The sequence

(8–12) $bb, baab$

is not a derivation since $baab$ does not follow from bb by the rules of P. This does not necessarily mean that the string $baab$ is not a theorem since there may exist some derivation in the system in which $baab$ is the last line. It happens in this case that there is, viz ,

(8–13) $aa, baab$

and thus $baab$ is a theorem.

One consequence of the definition is that the first line of a derivation must be an axiom since there are no lines preceding the first from which it could follow. Thus, a sequence such as

(8–14) $ab, aaba, baabab$

is not a derivation because ab is not an axiom. A derivation may, however, consist of only one line and, if so, that line must necessarily be an axiom.

The set of productions P in (8-8) is an infinite set of all ordered pairs of the form (x, axa) and (x, bxb), where x is a variable whose values are all the strings in A^*. P, therefore, contains productions such as L(a, aaa) and $(ab, babb)$ that will never actually be used in the derivation of any theorems in this system from the given set of axioms Further, because x is a variable symbol and not a member of the alphabet A, the expressions (x, axa) and (x, bxb) are not themselves productions but rather *production schemata* or formulas for constructing productions This finite set of schemata specifies an infinite set of productions in which the variable symbol x is replaced by any constant string on A^*. To be completely formal, we could, of course, give a recursive definition of the set of productions, thus embedding one recursive specification within another as we did in (8-5) and (8-7).

The axioms may also be specified by schemata containing variable symbols (or by recursive definition). For example, in the axiomatic system given in (8-15), whose theorems are all the *wff*'s of statement logic, S is an axiom schema specifying as an axiom any string consisting of the symbol p followed by any number of primes [cf. (8-7)]. P is also a schema for the infinite set of productions of this system.

(8–15) $A = \{\&, \vee, \sim, \rightarrow, \leftrightarrow, (,), p, '\}$
$S = \{px \mid x \in \{'\}^*\}$
$P = \{(x, \sim x), (x, y, (x \,\&\, y)), (x, y, (x \vee y)), (x, y, (x \leftrightarrow y))\}$ where x and y are strings in A^*

Problem: Which of the following sequences are derivations in the axiomatic system of (8-15)?

1. p, $\sim p$, $\sim\sim p$
2. p, p', $(p \vee p')$, $((p \vee p') \,\&\, p'')$
3. $(p \vee p)$, p', $(p' \rightarrow (p \vee p))$
4. p, $\sim p$, p'