8.5.8 Axiomatization of set theory

The primitives of set-theory are of course the notions 'set' and 'member'. What are the axioms of set-theory, the assumptions from which we may derive all we know about sets and their members? There are a number of different axiomatizations, characterizing distinct set-theories, but the best known one, which we give here, is known as the Zermelo-Frankel axiomatization (abbreviated ZF). This axiomatization appears to be quite successful in that its axioms are very intuitive, simple truths about sets, and no contradictions can be derived from them. One axiom, the axiom of *extensionality*, says that a set is uniquely determined by its members The other axioms either state that a certain set exists or that a certain set can be constructed by application of an operation. These axioms provide the foundation from which we may derive theorems about sets or set-theoretic concepts and, for instance, prove the exact relationships between properties of relations, and properties of their inverses or complements. Notationally we do not distinguish between sets and members, as we did with upper and lower case letters in the previous chapters, i.e , w, x, y, z are arbitrary set-theoretic objects, but anything enclosed in braces is a set The membership relation holds between a member and the set it is a member of, but $x \in x$ is not excluded

(8–41) The Zermelo-Frankel Axioms of Set Theory

Axiom 1. Extensionality *If x and y have the same elements, $x = y$.*

Axiom 2. Regularity *For every non-empty set x there is $y \in x$ such that $x \cap y = \emptyset$.*

Axiom 3. Empty set *There is a set with no members*

Axiom 4. Pairing *If x and y are sets, then there is a set z such that for all w, $w \in z$ if and only if $w = x$ or $w = y$.*

Axiom 5. Union *For every x there is a y such that $z \in y$ if and only if there is a $w \in x$ with $z \in w$.*

Axiom 6. Power set *For every x there is a y such that for all z, $z \in y$ if and only if $z \subseteq x$.*

Axiom 7. Infinity *There is a set x such that $0 \in x$ and whenever $y \in x$, then $y \cup \{y\} \in x$.*

Axiom 8. Replacement *If P is a functional property and x is a set, then the range of P restricted to x is a set; i.e., there is a set y such that for every z, $z \in y$ if and only if there is a $w \in x$ such that $P(w) = z$.*

The axiom of Regularity says that, if we are collecting objects into sets, we may stop at any stage and what we have then collected is a set. It is perhaps not really 'self-evident' that this is true, but at least it can be proven to be consistent with all the other axioms, and it is a very powerful axiom in constructing simple and direct proofs of other theorems. The Empty-set axiom implies together with Extensionality that there is exactly one empty set. Pairing guarantees that for every x and y the set $\{x, y\}$

exists Union and Power-set assert existence of these sets formed from arbitrary x. Infinity proves to be essential in representing the natural numbers as sets. Replacement is the one axiom that Frankel added to Zermelo's axiomatization, instead of his axiom of Separation, which says that a definable subset of a set is also a set, i.e. if x is a set and P is a property then there is a subset y of x which contains just the elements of x which have property P. Separation follows from Replacement, but Replacement does not follow from Separation. There are also statements which cannot be proved from the axioms 1-7 with Separation, but which are provable from 1-7 with Replacement.

These axioms are sufficient as foundations of mathematics, and note that the only primitive relation is membership Yet there are statements which cannot be proved or disproved from this axiomatization. One in particular is often assumed as an additional axiom: the Axiom of Choice. Let A be a set of non-empty sets. A *choice-function* for A is a function F with domain A and $F(X) \in X$ for each $X \in A$. The function F "chooses" an element in each $X \in A$, namely $F(X)$.

(8–42) **Axiom of Choice** Every set of non-empty sets has a choicefunction.

This axiom is often used in set theory, and has a variety of guises. It is not provable from axioms 1-8 as Paul Cohen proved in 1963; it is consistent with them, and no contradiction is derivable from it with 1-8, which Gödel proved in 1938. Yet its acceptance is not universal, and there are theorems which admit of simpler proofs with it but which also have more complicated proofs without using the Axiom of Choice. The results of Gödel and Cohen are milestones in the foundations of mathematics, producing innovative and fruitful proof techniques with wide new applications. For our present purposes it suffices to know that the Axiom of Choice is not universally accepted and granted equal status with the other axioms of set theory, although in the sequel we will implicitly rely on it as an additional axiom. (Axioms 1-8 + the Axiom of Choice are abbreviated to ZFC.)