8.5.7 Models for Peano's axioms

Peano's axioms, repeated below, were introduced in section 8.4, where we showed their connection to the important concept of proof by mathematical induction. In this section we return to them from a semantic perspective, to consider some of their models in addition to the intended model, the natural numbers. In Section 8.6.7 below we will discuss the relation of Peano's famous fifth axiom to the well-ordering axiom mentioned in 8.5.5. The first four axioms are first-order; the fifth is not.

(8–40) *Peano's axioms.* There are two primitive predicates, N and S. (The intended interpretation of N is 'is a natural number' and that of S is 'is the (immediate) successor of'.) There is one primitive constant, 0, whose intended interpretation is the natural number zero.

P1) $N0$

P2) $\forall x(Nx \to \exists y(Ny \,\&\, Syx \,\&\, \forall z(Szx \to z = y)))$

P3) $\sim \exists x(Nx \,\&\, S0x)$

P4) $\forall x\forall y\forall z\forall w((Nx \,\&\, Ny \,\&\, Szx \,\&\, Swy \,\&\, z = w) \to x = y)$

P5) If Q is a property such that

(a) $Q0$

(b) $\forall x\forall y((Nx \,\&\, Qx \,\&\, Ny \,\&\, Syx) \to Qy)$,

then $\forall x(Nx \to Qx)$

Peano, like Euclid, conceived of the primitive terms of the system as already having known meaning, and of the axioms as the smallest set of true statements about the natural number series from which its other properties could be derived. But if we look at the system in the purely formal way described above, we find that other meanings can be given to the primitives, and each of these interpretations would impart another meaning to derived statements about the natural numbers. Russell gives some instructive examples:[3]

(1) Let '0' stand for 100 and let 'natural number' be taken to mean the integers from 100 onward. All the axioms are satisfied, even the third; for although 100 is ordinarily the successor of 99, 99 is not a 'natural number' in this interpretation

(2) Let '0' be 0 but let 'natural number' be interpreted as 'even number'

[3]This is part of an interesting discussion in Waismann, Chapter 9. See also his Chapter 6.

and let the 'successor' of a number be that number obtained by adding 2 to it. The number series will now read

$$0, 2, 4, 6, 8, \ldots$$

and again all five of Peano's axioms are satisfied

(3) Let '0' be 1, let 'natural number' be any number of the sequence

$$1, 1/2, 1/4, 1/8, 1/16, \ldots$$

and let 'successor of' mean 'half of'. All five axioms also hold on this interpretation.

By contrast, we might consider some interpretations of N and S which do *not* satisfy all five Peano axioms.

(4) Let '0' stand for 0, 'successor' for successor, and let 'natural number' be interpreted as 'natural number less than or equal to 100'. Then axioms P1, P3, P4, and P5 hold, but P2 does not, because 100 does not have a successor in this interpretation. Similarly, *no* finite set can satisfy all of the Peano axioms.

(5) Let '0' stand for 0, let the 'successor' of any number be the number gotten by adding 1 to it (as in the standard interpretation), but let the 'natural numbers' be 0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, ... Axioms P1, P2, P3, and P4 hold; for instance, the unique successor of 1.5 is 2.5, and of 1 is 2; the unique predecessor of 7.5 is 6.5, and of 8 is 7. No fractional number is the successor of any whole number, and vice versa. The only axiom violated by this interpretation is P5, the induction principle. A property Q could satisfy (i) and (ii) of P5 and still fail to hold of all the 'natural numbers' by failing to hold for 0.5, 1.5, 2.5, ..., which will be missed by the "domino attack" of (i) and (ii).