Chapter 4

Infinities

In the preceding chapters we have occasionally dealt with sets, such as the set of positive integers, which we intuitively regard as infinite. We now want to examine the concept of infinity in more detail.

Some initially plausible approaches to the problem of characterizing infinity are not satisfactory. A definition employing the terms 'never-ending' or 'impossible, in principle, to list exhaustively,' for example, would be defective, since these expressions are themselves no clearer than the term 'infinite' that is to be explicated. What is needed is a definition that makes use of set-theoretic concepts already at hand and that accords with our intuitions about what sets should be regarded as infinite. Since an infinite set is in some sense "larger" than any finite set, we start by defining what it means for two sets to be of equal or unequal size.

4.1 Equivalent sets and cardinality

We say that two sets A and B have the same number of members, or are *equivalent*, if and only if there exists a one-to-one correspondence between them. Since a one-to-one correspondence is a function that is one-to-one and onto, every member of A is paired with exactly one member of B, and vice versa. In such a situation it would certainly be reasonable to say that the sets are of equal size. We denote the equivalence of A and B by $A \sim B$

The terms *equal* and *equivalent* must not be confused. Equal sets have *the same members* while equivalent sets have *the same number of members*. Equal sets, are therefore, necessarily equivalent but the converse is, in general, not true. Further, nothing is said in the definition of equivalence about the exact nature of the one-to-one correspondence between the sets – only that one exists.

For the case of finite sets this definition of equivalence leads to the expected conclusion. A set with just four distinct members, for example, can be put into one-to-one correspondence with any other set having exactly four distinct members, but not with any set with more or fewer members. The relation of equivalence of sets is, as the name implies, an equivalence relation with the property that all of the sets with the same number of members are put into the same equivalence class. To each equivalence class we can assign a number, called the *cardinal number*, denoting the size of each set in the class. For finite sets, the cardinal numbers correspond exactly to the natural numbers. Thus a set A with just four members is said to have a *cardinality* of 4, written $|A| = 4$, as we indicated in Chapter 1.

In the case of infinite sets something rather surprising happens. Consider, for example, the set of positive integers P, the set E of positive even integers (without zero), and the function F from P to E that maps every integer x into $2x$ as indicated in Figure 4-1.

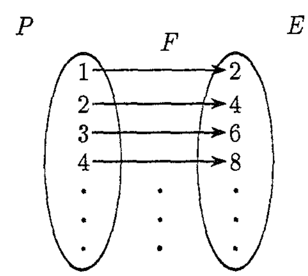


Figure 4–1: A one-to-one mapping from the positive integers to the positive even integers.

Every positive integer can be multiplied by 2 to give as a unique value a positive even integer. This shows that F is a function whose range is in E. The function F is one-to-one because for any integers x and y, if $2x = 2y$, then $x = y$. Further F is onto, since every member of E can be represented as $2x$, for some positive integer x. Thus, F is a one-to-one correspondence, and P and E, being equivalent sets, have the same number of members. This result is surprising in view of the fact that E is a proper subset of P (3, for example, is in P but not in E). We are accustomed to thinking of a set as being "larger" than any of its proper subsets, but if we adopt the notion of equivalence as the criterion for equal size of sets, then we are inescapably led to conclude that sometimes a set and a proper subset of that set may have the same number of members. If, on the other hand, we were to say that a set is always "larger" than a proper subset of itself, we would have to accept the puzzling consequence that sets of different size can be put into one-to-one correspondence. Either way the situation seems paradoxical When we examine the sets that exhibit this unusual behavior, however, we find that they are just the ones that we would intuitively call infinite. Accordingly, we define an infinite set in the following way:

DEFINITION 4.1 *A set is* infinite *iff it is equivalent to a proper subset of itself.* ∎

(4–1) *Example:* The set of natural numbers $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$ is infinite. Consider the set $P = \{1, 2, 3, 4, \ldots\}$, which is a proper subset of $\mathbf{N}$ and establish the mapping indicated in Figure 4-2 in which each natural number n is carried into $n + 1$. To each member of $\mathbf{N}$ there corresponds a unique member of P, and vice versa. Therefore, G is a one-to-one correspondence, and $P \sim \mathbf{N}$.

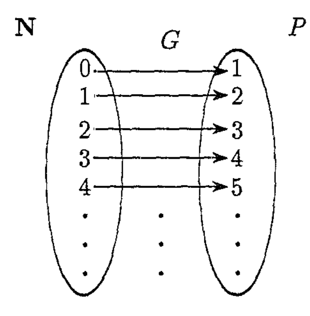


Figure 4–2: Mapping showing that the set $\mathbf{N}$ is equivalent to a proper subset of itself

(4–2) *Example:* The set of all (finite) strings A^* on the alphabet $\{a, b\}$ is infinite. Take as a proper subset of A^* the set $B = \{b, ba, bb, baa, bab, bba, \ldots\}$ i.e., all strings in A^* beginning with b. The mapping h shown in Figure 4-3 is a one-to-one correspondence because for every string x in A^* there is a unique string bx in B, and vice versa (e is the empty string of zero length).

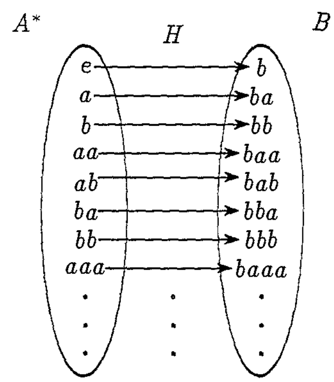


Figure 4–3: A one-to-one mapping of $\{a, b\}^*$ onto a proper subset of itself.

It should be easy to see that no finite set can be equivalent to one of its proper subsets (take, for example, the set $\{a, b, c\}$ and any of its proper subsets). One point about the definition of *infinite sets* sometimes causes confusion: Only the *existence* of at least one equivalent proper subset is required. The definition does not say that an infinite set is equivalent to *every* proper subset of itself, a condition that in fact could never be met. For example, $\mathbf{N}$ is not equivalent to its proper subset $\{0, 3, 18\}$.

4.2 Denumerability of sets

We have said that we can associate with each finite set a natural number that represents its cardinality, and that sets with the same cardinality form an equivalence class. Equivalent infinite sets can also be grouped into equivalence classes, all members of which have the same cardinality, but there is no positive integer that can be associated uniquely with such an equivalence class as its cardinal number. This follows from the fact that every integer is the cardinal number of a class of finite sets, and no infinite set can be equivalent to a finite set, since no one-to-one correspondence between them is possible. Nonetheless, it is convenient to have symbols denoting the cardinality of infinite sets; the one conventionally adopted as the cardinal number of the set of natural numbers (and all sets equivalent to it) is $\aleph_0$ (aleph null or aleph zero). It must be emphasized as we have said, that $\aleph_0$ is not a natural number, i.e., not a member of the set $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$. Each natural number has a corresponding cardinal number, but there are cardinal numbers, e.g. $\aleph_0$ that correspond to no natural number. A cardinal number can be regarded as an answer to a question about the number of members in a set. If we ask 'How many natural numbers are there?' or 'How many positive integers are there?', the answer is the cardinal number $\aleph_0$

By definition, a set with cardinality $\aleph_0$, i.e., one that is equivalent to the set of natural numbers, is called *denumerable* or *denumerably infinite* or *countably infinite*. A set that is either finite or denumerably infinite is called *countable*. We have already seen that the set of positive even integers (E in Figure 4-1) is denumerable. Here are some other examples:

(4–3) *Example:* The set of integers, including zero, $\mathbf{Z} = \{0, +1, -1, +2, -2, +3, -3, \ldots\}$, is denumerably infinite. One possible one-to-one correspondence with $\mathbf{N}$ is

$$\begin{array}{rcccccccccc} \mathbf{Z} & = & \{0, & +1, & -1, & +2, & -2, & +3, & -3, & \ldots\} \\ & F & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \mathbf{N} & = & \{0, & 1, & 2, & 3, & 4, & 5, & 6, & \ldots\} \end{array}$$

The function $F: \mathbf{Z} \rightarrow \mathbf{N}$ is defined by

$$F(x) = \begin{cases} 0 & \text{when } x = 0 \\ 2x - 1 & \text{when } x \text{ is positive} \\ -2x & \text{when } x \text{ is negative} \end{cases}$$

That F is indeed a one-to-one correspondence can be seen by noting that positive numbers in $\mathbf{Z}$ correspond to odd numbers in $\mathbf{N}$, and negative numbers in $\mathbf{Z}$ correspond to even numbers in $\mathbf{N}$ (with 0 corresponding to 0).

(4–4) *Example:* The set of reciprocals of the natural numbers without zero $S = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\}$ is denumerably infinite, as shown by the following one-to-one correspondence with $\mathbf{N}$:

$$\begin{array}{rcccccccc} S & = & \{\frac{1}{1}, & \frac{1}{2}, & \frac{1}{3}, & \frac{1}{4}, & \frac{1}{5}, & \frac{1}{6}, & \ldots\} \\ & G & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \mathbf{N} & = & \{0, & 1, & 2, & 3, & 4, & 5, & \ldots\} \end{array} \qquad G(x) = \tfrac{1}{x} - 1$$

(4–5) *Example:* The set of odd positive integers $F = \{1, 3, 5, 7, 9, \ldots\}$ is denumerably infinite. One possible one-to-one correspondence with $\mathbf{N}$ is

$$\begin{array}{rccccccc} F & = & \{1, & 3, & 5, & 7, & 9, & \ldots\} \\ & H & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \mathbf{N} & = & \{0, & 1, & 2, & 3, & 4, & \ldots\} \end{array} \qquad H(x) = \tfrac{x-1}{2}$$

We have seen that the set of positive integers P, the set of even positive integers E, and the set of odd integers F all have the same cardinality. Since $P = E \cup F$ one might have supposed that P would have more members than either E or F, but this is not the case. Thus, the union of two infinite sets is not necessarily a set with greater cardinality.

Are there sets larger than the set of positive integers? One that might intuitively seem so is the set of ordered pairs in the Cartesian product $\mathbf{N} \times \mathbf{N}$. When the pairs are listed in the order indicated by the arrow in Figure 4-4, however, we find that the following one-to-one correspondence between $\mathbf{N} \times \mathbf{N}$ and $\mathbf{N}$ can be established, although in this case it is somewhat more difficult to prove that the correspondence is actually one-to-one.

One would also tend to think that there are more rational numbers than natural numbers, since there are an infinite number of rational numbers between any two natural numbers (recall that a rational number is one which can be represented as the ratio of two integers x/y where $y \neq 0$). However, a one-to-one correspondence can be established, proving that the sets are actually of the same cardinality.

To set up a correspondence, we write down the positive rational numbers in an array of the following form:

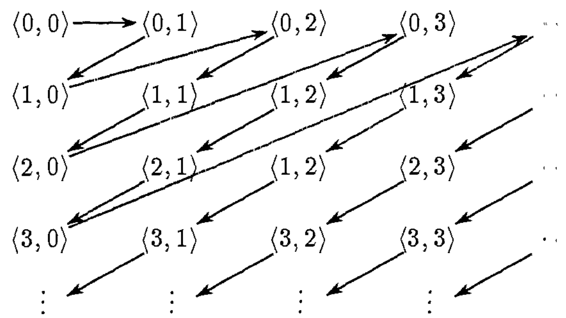


Figure 4–4: An enumeration of the members of $\mathbf{N} \times \mathbf{N}$.

$$\begin{array}{rcl} \mathbf{N} \times \mathbf{N} & = & \{ \langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 0,2\rangle, \langle 1,1\rangle, \langle 2,0\rangle, \langle 0,3\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \ldots \} \\ & F & \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \mathbf{N} & = & \{ 0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots \} \end{array}$$

Figure 4–5: A one-to-one correspondence between $\mathbf{N} \times \mathbf{N}$ and $\mathbf{N}$.

$1/1, 2/1, 3/1, 4/1, 5/1, 6/1, \ldots$

$1/2, 2/2, 3/2, 4/2, 5/2, \ldots$

$1/3, 2/3, 3/3, 4/3, \ldots$

$1/4, 2/4, 3/4, \ldots$

$1/5, 2/5, \ldots$

$1/6, \ldots$

$\vdots$

We first set up a correspondence between the elements of this array and the positive integers as follows: starting in the upper left-hand corner, count down the successive diagonals from the top row to the leftmost column. The first few terms of this correspondence are: $1/1$ to 1, $2/1$ to 2, $1/2$ to 3, $3/1$ to

4, 2/2 to 5, 1/3 to 6, 4/1 to 7, ..., etc. This is similar to the enumeration we gave in Figure 4-4. Next we pair the negative rationals with negative integers and 0 with 0 to give a complete correspondence between the integers and the rationals. We then make use of the established correpondence between the natural numbers and the integers to obtain a correspondence between the natural numbers and the rationals. (The rational numbers will each have been written down more than once by this procedure; e.g., 1/2 will also appear as 2/4, 3/6, etc. But having shown a one-to-one correspondence between this larger set and the natural numbers, it is easy enough to go through the list striking out each occurrence of a rational number which has already appeared in another form, moving the succeeding terms higher up in the list to fill in the gaps.) Putting the members of a set in a one-to-one correspondence with the natural numbers by means of some well-defined procedure such as this one is sometimes called *effectively listing* the members of that set.

4.3 Nondenumerable sets

Not only is there a procedure for effectively listing the ordered pairs of integers, one can also effectively list the ordered triples, quadruples, etc., *i.e.*, the set of n-tuples for any given n. (*Problem:* Give a systematic method for listing the ordered triples of integers as a linear sequence.) Thus, a set with cardinal number greater than $\aleph_0$ will not be found by taking successive Cartesian products of $\mathbf{N}$. At one time it was suppposed that there were no sets with cardinality greater than $\aleph_0$, but Georg Cantor (1845-1918), the mathematician who developed a large part of the theory of sets, proved that for any set A, the power set of A always has greater cardinality than A. Thus, the power set of $\mathbf{N}$ will have cardinality greater than $\mathbf{N}$.

THEOREM 4.1 *(Cantor): For any set* A, $|A|<|\wp(A)|$ ■

Proof: There is a function from $\wp(A)$ to A that maps every set containing just one element into that element in A, and maps all the other sets into some fixed element of A. This function is onto since every member of A has at least one correspondent in $\wp(A)$. Thus $|A|<|\wp(A)|$ or $|A|=|\wp(A)|$, *i.e.*, $\wp(A)$ is at least as large as A. We next show that there is no one-to-one and onto function F from A to $\wp(A)$, and thus that the sets cannot be equivalent. Assume that there is such an $F : A \to \wp(A)$. Then every member of A is mapped onto some subset of A. In general, some members of A will be mapped into a subset of which they are also members, and some will not. In the example in Fig. 4-6, 0 and 2 are each mapped by F into a set which

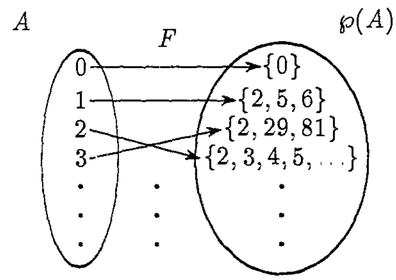


Figure 4–6: Illustration of an alleged one-to-one correspondence between a set and its power set.

has that element as a member, but 1 and 3 are not. Now form the set B by taking every member of A that is mapped into a subset not containing that member. That is, $B = \{x \in A \mid x \notin F(x)\}$. B is some subset of A and is therefore one of the members of $\wp(A)$ By hypothesis, F is onto, so there is at least one member of A that is mapped into B. Call this member y. Now we ask whether y is in B or not.

1. if $y \in B$ then it is not a member of the set it is mapped into, B. Thus if $y \in B$ then $y \notin B$. Contradiction!
2. if $y \notin B$ then it is one of those elements not in the set it is mapped into, so by definition it must be in B. So if $y \notin B$, then $y \in B$. Contradiction again! ■

This two-fold contradiction, which is reminiscent of Russell's Paradox (see Chapter 1 2), shows that the assumption that F is one-to-one and onto is false. Therefore it cannot be the case that $|A| = |\wp(A)|$, so we conclude that $|A| < |\wp(A)|$ A corollary of this important theorem is that there is a cardinal number greater than aleph-zero, which is commonly called $2^{\aleph_0}$, by analogy with the finite cardinals, where the power set of a set with n members has 2^n

members. $2^{\aleph_0}$ does not denote an integer or any other real number, however, since raising 2 to the power $\aleph_0$ is not a meaningful arithmetic operation.

Forming the power set of $\wp(\mathbf{N})$ leads to a cardinal number $2^{2^{\aleph_0}}$ that is larger than $2^{\aleph_0}$; $\wp(\wp(\wp(\mathbf{N})))$ has cardinality $2^{2^{2^{\aleph_0}}}$, and so on. Cantor's Theorem thus yields an infinite sequence of ever greater infinite cardinal numbers: $\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < \ldots$.

Another example of a nondenumerable set is the set of all real numbers between 0 and 1 (including 0 and 1 themselves), which we denote $[0,1]$. The real numbers consist of (1) the integers, (2) the other rational numbers and (3) the *irrational* numbers such as $\sqrt{5}$, π, $\frac{1}{3}\sqrt[3]{2}$, etc., which are not expressible as the ratio of two integers. In number theory it is proved that all real numbers, whether rational or irrational, can be written as an integer (possibly 0) followed by an infinitely long decimal fraction to the right of the decimal point. The fraction $\frac{1}{3}$, for example, can be written as 0.3333333 ..., where the ellipsis indicates that the sequence of 3's is infinite. Fractions such as $\frac{1}{2}$ can be represented as 0.5 or 0.50 or 0.500, etc., or else as the infinite repeating decimal 0.499999 ... Proof of this last statement would require an excursus into geometric series, but it can be made at least more plausible by considering the following: $\frac{1}{9} = 0.11111\ldots$; $1 = 9(\frac{1}{9}) = 9(0.11111\ldots) = 0.99999\ldots$ The decimal fraction of an irrational number is also infinitely long, but unlike a rational number it does not have repeating digit sequences.

Cantor's proof of the nondenumerability of $[0,1]$ begins with the assumption that every number in this set is uniquely represented by a sequence composed of 0 and an infinitely long decimal fraction. To assure that this representation is unique for each member of the set, we also take every rational number that might be written with an infinite string of 0's, e.g., 0.5000 ..., in the form having an infinite string of 9's, e.g., 0.4999 .... We now make the assumption that is to be proved false, namely, that the set $[0,1]$ is denumerable. If so, then its members can be put into a linear sequence with a first member, etc., and this sequence will contain every member of $[0,1]$. In Figure 4-7, this sequence $x_1, x_2, x_3, \ldots, x_n, \ldots$, is indicated as running vertically down the page with the decimal representation of each x_i to the right of the equals sign. The a's are the individual digits in each decimal fraction; a_{13}, for example, is the third digit in the decimal part of the first number in the sequence.

We now show that there is a number y in the set [0,1] that is not in the sequence $x_1, x_2, x_3, \ldots, x_n, \ldots$ This number has the following characteristics: the integer part is 0; the first decimal digit, a_{y1}, is different from

$$\begin{array}{lcl} x_1 & = & 0.a_{11}a_{12}a_{13}\cdots a_{1n}\cdots \\ x_2 & = & 0.a_{21}a_{22}a_{23}\cdots a_{2n}\cdots \\ x_3 & = & 0.a_{31}a_{32}a_{33}\cdots a_{3n}\cdots \\ \vdots & & \vdots \\ x_n & = & 0.a_{n1}a_{n2}a_{n3}\cdots a_{nn}\cdots \\ \vdots & & \vdots \end{array}$$

Figure 4–7: Putative enumeration of [0,1]

a_{11}; its second decimal digit, a_{y2}; is different from a_{22}; and in general the n^{th} decimal digit a_{yn} is different from a_{nn}. Therefore, y cannot be equal to x_1 because they differ in the first decimal place (and we have agreed that each number has a unique representation in the array); likewise, y cannot be equal to x_2 because they differ in the second decimal place; and in general, y cannot equal any number x_n in the array because it differs from y in (at least) the *nth* decimal place. Yet y is a number between 0 and 1 because it is of the form $y = 0.a_{y1}a_{y2}a_{y3}\cdots a_{yn}\cdots$. Thus, our assumption that the elements of [0,1] can be put into a linear sequence cannot be maintained, and the set is nondenumerable. This particular form of *reductio ad absurdum*, the so-called *diagonal argument* (y is constructed to be distinct from the integer $0.a_{11}a_{22}a_{33}\cdots a_{nn}\cdots$ on the diagonal of the square array), is encountered frequently in proofs involving infinite sets.

This proves that the cardinality of the set $[0,1]$ is greater than $\aleph_0$ but does not determine just what it is. Cantor was able to show (by a proof we will not reproduce here) that $[0,1]$ is equivalent to the power set of the integers, and thus its cardinal number is $2^{\aleph_0}$. Other sets with this cardinality are the set of all real numbers, the set of all points on a line (of whatever length), the set of all points on a plane, the set of all points in n-dimensional space (for any finite n), and the set of all subsets of the integers.

A problem that remained unsolved for many years was whether there are any infinite cardinal numbers other than $\aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}$, etc. Is there, for example, a cardinal number β such that $\aleph_0 < \beta < 2^{\aleph_0}$ or, to put it another way, is there a set intermediate in size between $\mathbf{N}$ and $\wp(\mathbf{N})$? The conjecture that the answer to this question was negative is known as the *Continuum Hypothesis*. It was not until 1963 that the matter was finally resolved (an event sufficiently newsworthy that it was reported in the *New York Times* (Nov. 14, 1963, p. 37)), when P.J. Cohen showed that the

Continuum Hypothesis can be neither proved nor disproved on the basis of the usual assumptions about set theory The Continuum Hypothesis is therefore *independent*, and either it or its negation could be added to set theory without being redundant or creating a contradiction.

The following examples further illustrate the diagonal method and some other methods of showing that a set has cardinality greater than $\aleph_0$.

(1) *The set of all real numbers x, $0 \leq x < 1$, written in binary notation.* The diagonal method can be applied to this set exactly as to the set of real numbers between 0 and 1 in decimal notation. Since every digit is either a 0 or a 1, one simply sets $y_{nn} = 1$ if $a_{nn} = 0$, and $y_{nn} = 0$ if $a_{nn} = 1$. The only reason for giving special mention to the binary notation case is that it is often easier to relate other sets to the real numbers in binary notation than to the real numbers in decimal notation

(2) *The set of all subsets of the set of natural numbers*, i.e., $\wp(\mathbf{N})$. For this example, we will use a method which is not overtly "diagonal", although it is closely related. (We already know from Theorem 4-1 that this set has cardinality greater than $\aleph_0$; we use the example to illustrate a method of proof.)

Assume that $\wp(\mathbf{N})$ has the same cardinality as the natural numbers, i.e. $\aleph_0$ Then it would be possible to list all the members of $\wp(\mathbf{N})$, i.e. all the subsets of $\mathbf{N}$, in some linear order, as $S_0, S_1, S_2, \ldots$ Suppose that we had a complete list of this sort. We could then construct a new subset of $\mathbf{N}$, to be called S^*, as follows:

> Let the natural number 0 be a member of S^* if and only if 0 is not a member of S_0
>
> Let $1 \in S^*$ if and only if $1 \notin S_1$.
>
> Let $2 \in S^*$ if and only if $2 \notin S_2$
>
> In general, let $n \notin S^*$ if and only if $n \notin S_n$.

Then S^* is a set of natural numbers, i.e., a subset of $\mathbf{N}$, which is different from each subset in the list by at least one member. If $n \in S_n$ for all n, then $S_0 = \emptyset$, and $\emptyset$ was not in the list Therefore the list could not have been complete after all, and the cardinality of $\wp(\mathbf{N})$ must be greater than $\aleph_0$.

(3) *The set of all languages on a finite alphabet.* Given an alphabet $V = \{a_0, a_1, a_2, \ldots a_n\}$, define a *sentence* on V to be any finite string of

elements of V (allowing repetitions). Define a *language* on V to be any set of sentences on V.

As a preliminary step, we will show that the set of all *sentences* on V has cardinality $\aleph_0$, by showing how the sentences can be listed in a single linear list We will list first all the 1-symbol sentences, and then all the 2-symbol sentences, etc. Within each group, the sentences can be listed in alphabetical order, letting a_0 be the first symbol and a_n the last. Thus the list will begin as follows:

$$\begin{array}{l} a_0 \\ a_1 \\ \vdots \\ a_n \\ a_0a_0 \\ a_0a_1 \\ \vdots \\ a_0a_n \\ a_1a_0 \\ a_1a_1 \\ \vdots \\ a_1a_n \\ a_2a_0 \\ \vdots \\ a_na_n \\ a_0a_0a_0 \\ \vdots \\ a_na_na_n \\ a_0a_0a_0a_0 \\ \vdots \end{array}$$

Since all the sentences are clearly included in the list, they can be numbered $0, 1, 2, \ldots$, thus establishing a one-one correspondence between the set of sentences and the natural numbers.

Having established that the set of all *sentences* on V has cardinality $\aleph_0$, we can now show that the set of all *languages* on V has a greater cardinality. We will show three different methods of proof which can be used.

(i) (Diagonal proof.) Assume that the set of all languages on V has cardinality $\aleph_0$, so that the languages can be listed $L_0, L_1, L_2, \ldots$. We have already established a means of listing all the sentences on V as $s_0, s_1, s_2, \ldots$. Then we can set up an infinite square array of 0's and 1's as shown below, where an entry x_i^k is 0 if s_i is not in L_k and x_i^k is 1 if s_i is in L_k.

	s_0	s_1	s_2	s_3	s_4	$\ldots$
L_0	x_0^0	x_1^0	x_2^0	x_3^0	x_4^0	$\ldots$
L_1	x_0^1	x_1^1	x_2^1	x_3^1	x_4^1	$\ldots$
L_2	x_0^2	x_1^2	x_2^2	x_3^2	x_4^2	$\ldots$
L_3	x_0^3	x_1^3	x_2^3	x_3^3	x_4^3	$\ldots$
L_4	x_0^4	x_1^4	x_2^4	x_3^4	x_4^4	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Thus for instance the language consisting of all and only the odd-numbered sentences would be represented by a row 010101...; the language consisting of all the 1-symbol sentences (a_0 through a_n) would be represented by a row whose first $n + 1$ entries were 1's, with all the remaining entries 0's.

Then we can construct a representation of a language L^* different from any in the list as follows: Let $x_0^* = 0$ if $x_0^0 = 1$; $x_0^* = 1$ if $x_0^0 = 0$. In the same way make x_1^* different from x_1^1, x_2^* different from x_2^2, etc.; in general, $x_m^* = 0$ if $x_m^m = 1$, and $x_m^* = 1$ if $x_m^m = 0$. Then by the given interpretation of 0's and 1's, it follows that s_m is in L^* if and only if s_m is not in L_m, and thus that L^* differs by at least one sentence from every language in the list. Since the procedure applies to any such putative list of all languages, it follows that there cannot be such a list, and therefore that the set of all languages on V has a cardinality greater than $\aleph_0$.

(ii) The second proof is analogous to the proof used for the set $\wp(\mathbf{N})$ given as example (2) above. Let S be the name of the set of all *sentences* on V. Then since every language on V is a set of sentences on V, and every set of sentences on V is a language on V, the set of all languages on V is exactly the set of all subsets of S, i.e. $\wp(S)$. Then suppose that the set of all languages on V had cardinality $\aleph_0$. We could then list all the languages, i.e. all the members of $\wp(S)$, in a single list, $L_0, L_1, L_2, \ldots$. But then we could immediately construct a new language L^* as follows (using the enumeration of the sentences of S previously established): let $s_0 \in L^*$ if and only if

$s_0 \notin L_0$, $s_1 \in L^*$ if and only if $s_1 \notin L_1$, etc.; in general, $s_m \in L^*$ if and only if $s_m \notin L_m$. Thus L^* is a subset of S which differs from every language in the list by at least one member, and the list, therefore, could not have been complete. Therefore, the set of all languages, $\wp(S)$, cannot have cardinality $\aleph_0$.

(iii) The third proof is an example of a general method: to show that a given set has cardinality greater than $\aleph_0$, it is sufficient to show that it can be put into a one-one correspondence with a set already known to have cardinality greater than $\aleph_0$. Since the set of real numbers between 0 and 1 in binary notation is already known to have cardinality greater than $\aleph_0$, we will set up a one-one correspondence between it and the set of all languages on V.

Let each language be represented as an infinite sequence of 0's and 1's in the manner described in the first method of proof above. (We do not, however, assume that the languages can be listed in a linear order, since we have already seen that such an assumption leads to a contradiction.) Then each language can be paired with a unique real number between 0 and 1, since the infinite decimal is also an infinite sequence of 0's and 1's designating exactly one language and exactly one real number.

The establishment of the correspondence completes the proof

The three methods of proof outlined above are equally valid The first two have the advantage of not requiring prior knowledge of any sets with cardinality greater than $\aleph_0$, but once such knowledge is at hand, the third method is often more convenient. Furthermore, only the third method, setting up a one-one correspondence, can establish exactly what the cardinality of a set is, and then only when the cardinality of the corresponding set is known. In the examples above, all the sets with cardinality greater than $\aleph_0$ have the same cardinality as the set of real numbers, but we have not proved the fact for any of the sets, and we cannot take it for granted because, as we have seen, there are in fact infinitely many different cardinalities greater than $\aleph_0$.

A set which is not countable is called *uncountable* or *non-denumerable* or *non-denumerably infinite*.

4.4 Infinite vs. unbounded

There is sometimes confusion over the difference between the terms 'infinite' and 'unbounded', particularly with respect to statements like 'The length of English sentences is unbounded', or 'English has sentences of unbounded length.' *Unbounded* means 'having no upper bound', i.e. having no limiting value such that every value is at or under that limit. Both of the cited sentences mean simply that there is no fixed length such that all English sentences are of that length or less, and this is perfectly consistent with the statement that every English sentence is finite in length. One can argue validly from the premise that the length of English sentences is unbounded to the conclusion that the *set* of English sentences is infinite (see problem 4 in the following exercises), but one cannot validly argue from that to the conclusion that the *length* of some English sentence is infinite.

Further examples

(1) The number of sides of regular polygons is unbounded, since for any polygon with n sides, there is another with $n + 1$ sides; but the number of sides is always finite. The *set* of such polygons is infinite.

(2) Consider the set of real numbers x such that $0 < x < 1$. Although there is no largest real number in that set (1 itself is excluded from the set, and for every real number less than 1, there is a larger real number that is less than 1), the size of the real numbers in that set is *bounded*, since 1 serves as an upper bound. In this case, the size of the members of the set is bounded, but the set itself is nevertheless infinite.

(3) Starting with the words in some given English dictionary, the length of English sentences that do not use any word more than once is bounded. (The number of distinct words in the given dictionary would provide an upper bound; it is irrelevant to the question of boundedness whether an English sentence of that length could actually be constructed.)

As can be seen from the examples, the terms 'bounded' and 'unbounded' apply to values of functions, or to measures of various sorts applied to members of a set; these terms do not describe cardinalities of sets, as do 'finite' and 'infinite'. It is never strictly meaningful to speak of an 'unbounded set', although such a phrase may sometimes be interpretable in context as elliptical for some longer phrase. Confusion can be most easily avoided by eschewing the use of the term 'unbounded' altogether, and replacing statements like the first two above by statements like 'There is no upper bound on the length of English sentences'. For the reader who encounters the term 'unbounded' in a statement, it may be advisable to ascertain whether the statement can be unambiguously recast in such a form before proceeding.

Exercises

1. Show that the relation of equivalence of sets is in fact an equivalence relation.

2. Show that the set of integral powers of 10 $\{10, 100, 1000, 10,000, 100,000, \ldots\}$ is denumerably infinite

3. Show that the set of all negative integers is infinite

4. Suppose that the following assumptions are true of English:

   (i) There is a finite alphabet for writing sentences, consisting of 26 letters, a set of punctuation marks and a space

   (ii) Every sentence is a finite string in the alphabet given in (i).

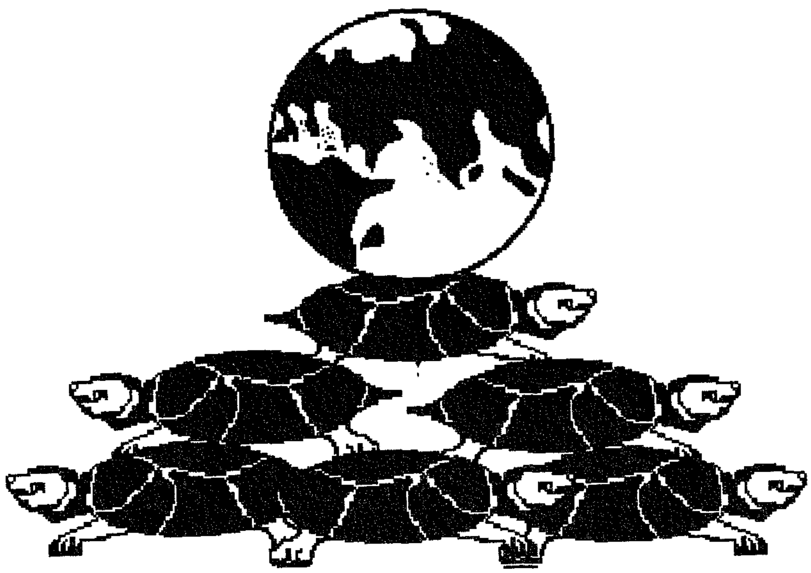
   (iii) There is no upper bound on the length of sentences of English. E.g. given any sentence, a longer one can be made by conjoining it with another one.

   What then is the cardinality of the set of all sentences of English? Motivate your answer

5. A hotelkeeper has a hotel with a denumerably infinite number of rooms, all single rooms, numbered $1, 2, 3, 4, 5 \ldots$ On Saturday night the hotel was full, but Joe Doe came in asking for lodging. The obliging hotelkeeper, using his intercom, asked each guest to move into the room $n + 1$ when his present room was numbered n. So Joe Doe was given room 1. But on Sunday everyone stayed for another night. Now a denumerably infinite football team came in asking for lodgings one room per person How could the obliging hotelkeeper accommodate them?

6. Assume that the earth rests on the back of a giant turtle, and that the turtle sits on the back of two giant turtles, and those two on three, etc. 'all the way down' (i.e. there is no bottom layer of turtles).[1]

[1] This problem was inspired by a legendary anecdote reported in the preface of an equally legendary, but actual Ph.D. dissertation, *Constraints on Variables in Syntax* by J. R. Ross, MIT 1967. Since only parts of the dissertation are published, we repeat the anecdote here as told by Ross for historically accurate preservation:



(a) Suppose each turtle is the sole deity of some monotheistic sect (exactly one sect per turtle). What is the cardinality of the set of all such sects?

(b) Suppose each *subset* of the set of all these earth-supporting turtles forms the deity-group of some one sect (a-, mono- or polytheistic, with the latter including both finite and infinite numbers of deities). What is the cardinality of the set of all such sects?

After a lecture on cosmology and the structure of the solar system, William James was accosted by a little old lady. "Your theory that the sun is the center of the solar system, and that the earth is a ball which rotates around it has a very convincing ring to it, Mr. James, but it's wrong. I've got a better theory", said the little old lady. "And what is that, madam?" inquired James politely. "That we live on a crust of earth which is on the back of a giant turtle". Not wishing to demolish this absurd little theory by bringing to bear the masses of scientific evidence he had at his command, James decided to gently dissuade his opponent by making her see some of the inadequacies of her position. "If your theory is correct, madam," he asked, "what does this turtle stand on?" "You are a very clever man, Mr. James, and that's a very good question" replied the little old lady, "but I have an answer to it. And it's this: the first turtle stands on the back of a second, far larger turtle, who stands directly under him". "But what does this second turtle stand on?" persisted James patiently. To this the little old lady crowed triumphantly. "It's no use, Mr. James - it's turtles all the way down."

(Note that two different sects may of course worship some turtles in common as long as they do not worship exactly the same set.)

7. Cardinal numbers form their own numerical system in which we can do *cardinal arithmetic*. This exercise gives the basic notions. Let A and B be disjoint sets, finite or infinite, and let $a = |A|$ and $b = |B|$. We define cardinal addition, written $\oplus$, and cardinal multiplication, written $\otimes$, as follows:

$$\begin{aligned} a \oplus b &= |(A \cup B)| \\ a \otimes b &= |(A \times B)| \end{aligned}$$

When A and B are both finite, cardinal addition and multiplication produce the same results as the corresponding arithmetic operations on integers. When at least one is infinite, however, the operations are no longer parallel in all respects. Find examples of sets A and B for which the following hold:

$$\begin{array}{lll} \text{(a)} & \aleph_0 \oplus 1 & = \aleph_0 \\ \text{(b)} & \aleph_0 \otimes 2 & = \aleph_0 \\ \text{(c)} & \aleph_0 \oplus \aleph_0 & = \aleph_0 \\ \text{(d)} & \aleph_0 \otimes \aleph_0 & = \aleph_0 \end{array}$$

Do the operations $\oplus$ and $\otimes$ appear to be commutative and associative?

8. It can be proved that $\aleph_0$ is the smallest infinite cardinal number. Consider the following putative counterexample to this claim. Choose a cardinal number x such that $2^x = \aleph_0$. x cannot be finite, since 2 raised to any finite power is finite; but x cannot be equal to $\aleph_0$ either, since $2^{\aleph_0} > \aleph_0$ by Cantor's Theorem. Therefore x is an infinite cardinal number less then $\aleph_0$. What is wrong with this argument?