8.5.2 Consistency, completeness, and independence

A formal system is *consistent* if it is not possible to derive from its axioms both some statement and the denial of that same statement. An inconsistent system cannot have a model, since no actual statement can be simultaneously true and false; hence one way to show that a system is consistent is to exhibit a model for it.

It is useful that we have both a syntactic and a semantic characterization of consistency known to be equivalent, since one is easier to apply in some cases and the other in others. In particular, when a system is inconsistent, it's usually easier to demonstrate that syntactically than semantically. That is, it's usually easier to derive a contradiction from the axioms than to prove by a meta-level argument that the system has no models Conversely, when a system is consistent, it's usually easier to show that semantically, by finding a model, than to demonstrate that it's impossible to derive a contradiction from the given axioms. When one doesn't know the answer in advance, it may be necesary to try both methods alternately until one of them succeeds.

The term *completeness* is used in various senses. What all notions of completeness have in common is that for a formal system to be complete in some sense, it must be possible to derive within the formal system all the statements satisfying some given criterion; different notions of completeness reflect different criteria for the desired statements. Among the most commonly encountered notions of completeness is one which is syntactic, since it is defined in terms of formal systems alone, and one which is semantic, defined in terms of the relations between formal systems and their models. A formal system is called *formally complete* if every statement expressible in the system, i.e. expressible using only the primitives of the system including a given formalized logic, can either be proved or disproved (its negation proved) in the system Other terms for the same or very similar notions are *deductively complete* (Copi), *complete with respect to negation* (Thomason), and *syntactically complete*. A formal system is *semantically complete with respect to a model M*, or *weakly semantically complete* (Thomason), if every statement expressible in the system which is true in the model M is derivable in the formal system.

The notion of *independence* concerns the question of whether any of the axioms are superfluous An axiom is *independent* if it cannot be derived from the other axioms of the system. A whole system is said to be *independent* (a slight abuse of language) if all of its axioms are independent. That was a syntactic characterization of independence (why?); a semantic chracterization is the following: A given axiom is independent of the other axioms of a system S if the system S′ that results from deleting that axiom has models which are not models of the whole system S In any reasonably "well-behaved" framework, the two notions of independence will be provably equivalent and one can use whichever one is easier to apply in a given case. As in the case of consistency, which is easier often depends on whether the answer is positive or negative. Determining precisely what it takes for a framework to be sufficiently "well-behaved" for the syntactic and semantic characterizations of independence to determine the same notion is one kind of question studied in the metamathematical side of model theory.

The three notions of consistency, completeness, and independence are not all of equal importance. Consistency is of fundamental importance, since it is obviously a minimal condition of adequacy on any set of axioms designed to formalize any system that is not meant to be self-contradictory. Completeness is often of theoretical importance to logicians, but (a) proving completeness for a system of any complexity generally requires a fairly high level of mathematical sophistication (and many important formal systems are provably incomplete); and (b) it is not obvious that completeness is ever any issue that linguists need to be concerned with. Questions of completeness will therefore be relatively neglected here. Independence of axioms is simply a matter of "elegance"; it is generally considered desirable in an axiom system, but has no significant consequences for the system as a whole.