and the product of two numbers are associative functions, since for all numbers x, y, and z we have: $(x + y) + z = x + (y + z)$ and $(x \times y) \times z = x \times (y \times z)$. The difference function is not associative: $(4 - 2) - 2 = 0$, but $4 - (2 - 2) = 4$. The associativity of a function f means that that we can write $x_1fx_2fx_3 \ldots x_{n-1}fx_n$ without having to insert any brackets, since the value of the expression is independent of where they are inserted. Thus, for example, we have: $(x_1 + x_2) + (x_3 + x_4) = x_1 + ((x_2 + x_3) + x_4)$. First one has $(x_1 + x_2) + (x_3 + x_4) = x_1 + (x_2 + (x_3 + x_4))$, since $(x + y) + z = x + (y + z)$ for any x, y, and z, so in particular for $x = x_1$, $y = x_2$, and $z = x_3 + x_4$. And $x_1 + (x_2 + (x_3 + x_4)) = x_1 + ((x_2 + x_3) + x_4)$, since $x_2 + (x_3 + x_4) = (x_2 + x_3) + x_4$.

2.5 The Semantics of Propositional Logic

The valuations we have spoken of can now, in the terms just introduced, be described as (unary) functions mapping formulas onto truth values. But not every function with formulas as its domain and truth values as its range will do as a valuation. A valuation must agree with the interpretations of the connectives which are given in their truth tables. A function which attributes the value 1 to both p and ¬p, for example, cannot be accepted as a valuation, since it does not agree with the interpretation of negation. The truth table for ¬ (see (14)) rules that for every valuation V and for all formulas ϕ:

(i) $V(\neg\phi) = 1$ iff $V(\phi) = 0$.

This is because the truth value 1 is written under $\neg\phi$ in the truth table just in case a 0 is written under ϕ. Since $\neg\phi$ can only have 1 or 0 as its truth value (the range of V contains only 1 and 0), we can express the same thing by:

(i′) $V(\neg\phi) = 0$ iff $V(\phi) = 1$.

That is, a 0 is written under $\neg\phi$ just in case a 1 is written under ϕ.

Similarly, according to the other truth tables we have:

(ii) $V(\phi \wedge \psi) = 1$ iff $V(\phi) = 1$ and $V(\psi) = 1$.
(iii) $V(\phi \vee \psi) = 1$ iff $V(\phi) = 1$ or $V(\psi) = 1$.
(iv) $V(\phi \rightarrow \psi) = 0$ iff $V(\phi) = 1$ and $V(\psi) = 0$.
(v) $V(\phi \leftrightarrow \psi) = 1$ iff $V(\phi) = V(\psi)$.

Recall that *or* is interpreted as *and/or*. Clause (iii) can be paraphrased as: $V(\phi \vee \psi) = 0$ iff $V(\phi) = 0$ and $V(\phi) = 0$; (iv) as: $V(\phi \rightarrow \psi) = 1$ iff $V(\phi) = 0$ or $V(\psi) = 1$ (*or* = *and/or*). And if, perhaps somewhat artificially, we treat the truth values 1 and 0 as ordinary numbers, we can also paraphrase (iv) as: $V(\phi \rightarrow \psi) = 1$ iff $V(\phi) \leq V(\psi)$ (since while $0 \leq 0$, $0 \leq 1$, and $1 \leq 1$, we do not have $1 \leq 0$).

A valuation V is wholly determined by the truth values which it attributes to the propositional letters. Once we know what it does with the propositions, we can calculate the V of any formula ϕ by means of ϕ's construction tree. If

$V(p) = 1$ and $V(q) = 1$, for example, then $V(\neg(\neg p \wedge \neg q))$ can be calculated as follows. We see that $V(\neg p) = 0$ and $V(\neg q) = 0$, so $V(\neg p \wedge \neg q) = 0$ and thus $V(\neg(\neg p \wedge \neg q)) = 1$. Now it should be clear that only the values which V attributes to the proposition letters actually appearing in ϕ can have any influence on $V(\phi)$. So in order to see how the truth value of ϕ varies with valuations, it suffices to draw up what is called a *composite truth table*, in which the truth values of all subformulas of ϕ are calculated for every possible distribution of truth values among the propositional letters appearing in ϕ. To continue with the same example, the composite truth table for the formula $\neg(\neg p \wedge \neg q)$ is given as (31):

(31)

	1	2	3	4	5	6
	p	q	¬p	¬q	¬p ∧ ¬q	¬(¬p ∧ ¬q)
V_1	1	1	0	0	0	1
V_2	1	0	0	1	0	1
V_3	0	1	1	0	0	1
V_4	0	0	1	1	1	0

The four different distributions of truth values among p and q are given in columns 1 and 2. In columns 3 and 4, the corresponding truth values of ¬p and ¬q have been given; they are calculated in accordance with the truth table for negation. Then in column 5 we see the truth values of $\neg p \wedge \neg q$, calculated from columns 3 and 4 using the truth table for conjunction. And finally, in column 6 we see the truth values of $\neg(\neg p \wedge \neg q)$ corresponding to each of the four possible distributions of truth values among p and q, which are calculated from column 5 by means of the truth table for negation.

The number of rows in the composite truth table for a formula depends only on the number of different propositional letters occurring in that formula. Two different propositional letters give rise to four rows, and we can say quite generally that *n* propositional letters give rise to 2^n rows, since that is the number of different distributions of the two truth values among *n* propositions. Every valuation corresponds to just one row in a truth table. So if we restrict ourselves to the propositional letters p and q, there are just four possible valuations: the V_1, V_2, V_3, and V_4 given in (31). And these four are the only valuations which matter for formulas in which p and q are the only propositional letters, since as we have just seen, what V does with ϕ is wholly determined by what V does with the propositional letters actually appearing in ϕ. This means that we may add new columns to (31) for the evaluation of as many formulas as we wish composed from just the letters p and q together with connectives. That this is of some importance can be seen as follows.

Note that the composite formula $\neg(\neg p \wedge \neg q)$ is true whenever any one of the proposition letters p and q is true, and false if both p and q are false. This is just the inclusive disjunction of p and q. Now consider the composite truth table given in (32):

(32)

	1	2	3	4	5	6	7
	p	q	$\neg$p	$\neg$q	$\neg$p $\wedge$ $\neg$q	$\neg$($\neg$p $\wedge$ $\neg$q)	p $\vee$ q
V_1	1	1	0	0	0	1	1
V_2	1	0	0	1	0	1	1
V_3	0	1	1	0	0	1	1
V_4	0	0	1	1	1	0	0

What we have done is add a new column to the truth table mentioned above in which the truth value of p $\vee$ q is given for each distribution of truth values among p and q, this being calculated in accordance with the truth table for the disjunction. This shows clearly that the truth values of $\neg(\neg p \wedge \neg q)$ and p $\vee$ q are the same under each valuation, since

$$V_1(\neg(\neg p \wedge \neg q)) = V_1(p \vee q) = 1;$$
$$V_2(\neg(\neg p \wedge \neg q)) = V_2(p \vee q) = 1;$$
$$V_3(\neg(\neg p \wedge \neg q)) = V_3(p \vee q) = 1;$$
$$V_4(\neg(\neg p \wedge \neg q)) = V_4(p \vee q) = 0.$$

So for every valuation V we have: $V(\neg(\neg p \wedge \neg q)) = V(p \vee q)$. The formulas $\neg(\neg p \wedge \neg q)$ and p $\vee$ q are (*logically*) *equivalent*. To put it more explicitly, ϕ and ψ are said to be (*logically*) *equivalent* just in case for every valuation V we have: $V(\phi) = V(\psi)$. The qualification *logical* is to preclude any confusion with *material equivalence*.

In order to see how all formulas of the form $\neg(\neg\phi \wedge \neg\psi)$ and $\phi \vee \psi$ behave under all possible valuations, a composite truth table just like (32) can be drawn up by means of the truth tables for negation, conjunction, and disjunction. The result is given in (33):

(33)

ϕ	ψ	$\neg\phi$	$\neg\psi$	$\neg\phi \wedge \neg\psi$	$\neg(\neg\phi \wedge \neg\psi)$	$\phi \vee \psi$
1	1	0	0	0	1	1
1	0	0	1	0	1	1
0	1	1	0	0	1	1
0	0	1	1	1	0	0

In this truth table it can clearly be seen that the equivalence of formulas of the form $\neg(\neg\phi \wedge \neg\psi)$ and $\phi \vee \psi$ is quite general (for a general explication of relationships of this sort, see theorem 13 in §4.2.2).

Consider another example. All formulas of the forms $\neg\neg\phi$ and ϕ are equivalent, as is apparent from (34):

(34)

ϕ	$\neg\phi$	$\neg\neg\phi$
1	0	1
0	1	0

This equivalence is known as the *law of double negation*. And the last example we shall give is a truth table which demonstrates that $(\phi \vee \psi) \vee \chi$ is equivalent to $\phi \vee (\psi \vee \chi)$, and $(\phi \wedge \psi) \wedge \chi$ to $\phi \wedge (\psi \wedge \chi)$; see (35):

(35)

ϕ	ψ	χ	$\phi \vee \psi$	$(\phi \vee \psi) \vee \chi$	$\psi \vee \chi$	$\phi \vee (\psi \vee \chi)$	$\phi \wedge \psi$	$(\phi \wedge \psi) \wedge \chi$	$\psi \wedge \chi$	$\phi \wedge (\psi \wedge \chi)$
1	1	1	1	1	1	1	1	1	1	1
1	1	0	1	1	1	1	1	0	0	0
1	0	1	1	1	1	1	0	0	0	0
1	0	0	1	1	0	1	0	0	0	0
0	1	1	1	1	1	1	0	0	1	0
0	1	0	1	1	1	1	0	0	0	0
0	0	1	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

The latter two equivalences are known as the *associativity of* $\wedge$ and the *associativity of* $\vee$, respectively, by analogy with the concept which was introduced in connection with functions and which bears the same name. (For a closer connection between these concepts, see §2.6.) Just as with functions, the associativity of $\vee$ and $\wedge$ means that we can omit brackets in formulas, since their meaning is independent of where they are placed. This assumes, of course, that we are only interested in the truth values of the formulas. In general, then, we shall feel free to write $\phi \wedge \psi \wedge \chi$, $(\phi \to \psi) \wedge (\psi \to \chi) \wedge (\chi \to \phi)$ etc. $\phi \wedge \psi \wedge \chi$ is true just in case all of ϕ, ψ, and χ are true, while $\phi \vee \psi \vee \chi$ is true just in case any one of them is true.

Exercise 6

A large number of well-known equivalences are given in this exercise. In order to get the feel of the method, it is worthwhile to demonstrate that a few of them are equivalences by means of truth tables and further to try to understand why they must hold, given what the connectives mean. The reader may find this easier if the metavariables ϕ, ψ, and χ are replaced by sentences derived from natural language.

Prove that in each of the following, all the formulas are logically equivalent to each other (independently of which formulas are represented by ϕ, ψ, and χ):

(a) $\phi, \neg\neg\phi, \phi \wedge \phi, \phi \vee \phi, \phi \wedge (\phi \vee \psi), \phi \vee (\phi \wedge \psi)$
(b) $\neg\phi, \phi \to (\psi \wedge \neg\psi)$
(c) $\neg(\phi \vee \psi), \neg\phi \wedge \neg\psi$ (*De Morgan's Law*)
(d) $\neg(\phi \wedge \psi), \neg\phi \vee \neg\psi$ (*De Morgan's Law*)
(e) $\phi \vee \psi, \psi \vee \phi, \neg\phi \to \psi, \neg(\neg\phi \wedge \neg\psi), (\phi \to \psi) \to \psi$
(f) $\phi \wedge \psi, \psi \wedge \phi, \neg(\phi \to \neg\psi), \neg(\neg\phi \vee \neg\psi)$
(g) $\phi \to \psi, \neg\phi \vee \psi, \neg(\phi \wedge \neg\psi), \neg\psi \to \neg\phi$
(h) $\phi \to \neg\psi, \psi \to \neg\phi$ (*law of contraposition*)
(i) $\phi \leftrightarrow \psi, (\phi \to \psi) \wedge (\psi \to \phi), (\phi \wedge \psi) \vee (\neg\phi \wedge \neg\psi)$
(j) $(\phi \vee \psi) \wedge \neg(\phi \wedge \psi), \neg(\phi \leftrightarrow \psi), \neg\phi \leftrightarrow \psi$, (and $\phi \infty \psi$, though officially it is not a formula of propositional logic according to the definition)
(k) $\phi \wedge (\psi \vee \chi), (\phi \wedge \psi) \vee (\phi \wedge \chi)$ (*distributive law*)
(l) $\phi \vee (\psi \wedge \chi), (\phi \vee \psi) \wedge (\phi \vee \chi)$ (*distributive law*)
(m) $(\phi \vee \psi) \to \chi, (\phi \to \chi) \wedge (\psi \to \chi)$
(n) $\phi \to (\psi \wedge \chi), (\phi \to \psi) \wedge (\phi \to \chi)$
(o) $\phi \to (\psi \to \chi), (\phi \wedge \psi) \to \chi$

The equivalence of $\phi \vee \psi$ and $\psi \vee \phi$ and of $\phi \wedge \psi$ and $\psi \wedge \phi$ as mentioned under (e) and (f) in exercise 6 are known as the *commutativity* of $\vee$ and $\wedge$, respectively. (For the connection with the commutativity of functions, see §2.6.) Both the equivalence mentioned under (h) and the equivalence of $\phi \to \psi$ and $\neg\psi \to \neg\phi$ given in (g) in exercise 6 are known as the *law of contraposition*.

Logically equivalent formulas always have the same truth values. This means that the formula χ' which results when one subformula ϕ of a formula χ is replaced by an equivalent formula ψ must itself be equivalent to χ. This is because the truth value of χ' depends on that of ψ in just the same way as the truth value of χ depends on that of ϕ. For example, if ϕ and ψ are equivalent, then $\phi \to \theta$ and $\psi \to \theta$ are too. One result of this is that the brackets in $(\phi \wedge \psi) \wedge \chi$ can also be omitted where it appears as a subformula of some larger formula, so that we can write $(\phi \wedge \psi \wedge \chi) \to \theta$, for example, instead of $((\phi \wedge \psi) \wedge \chi) \to \theta$, and $\theta \to ((\phi \to \psi) \wedge (\psi \leftrightarrow \chi) \wedge (\chi \vee \psi))$ instead of $\theta \to (((\phi \to \psi) \wedge (\psi \leftrightarrow \chi)) \wedge (\chi \vee \psi))$. More generally, we have here a useful way of proving equivalences on the basis of other equivalences which are known to hold. As an example, we shall demonstrate that $\phi \to (\psi \to \chi)$ is equivalent to $\psi \to (\phi \to \chi)$. According to exercise 6(o), $\phi \to (\psi \to \chi)$ is equivalent to $(\phi \wedge \psi) \to \chi$. Now $\phi \wedge \psi$ is equivalent to $\psi \wedge \phi$ (commutativity of $\wedge$), so $(\phi \wedge \psi) \to \chi$ is equivalent to $(\psi \wedge \phi) \to \chi$. Applying 6(o) once more, this time with ψ, ϕ, and χ instead of ϕ, ψ, and χ, we see that $(\psi \wedge \phi) \to \chi$ is equivalent to $\psi \to (\phi \to \chi)$. If we now link all these equivalences, we see that $(\phi \wedge \psi) \to \chi$ is equivalent to $\psi \to (\phi \to \chi)$, which is just what we needed.

Exercise 7 ◇

Show on the basis of equivalences of exercise 6 that the following formulas are equivalent:

(a) $\phi \leftrightarrow \psi$ and $\psi \leftrightarrow \phi$ (*commutativity of* $\leftrightarrow$)
(b) $\phi \to \neg\phi$ and $\neg\phi$
(c) $\phi \wedge (\psi \wedge \chi)$ and $\chi \wedge (\psi \wedge \phi)$
(d) $\phi \to (\phi \to \psi)$ and $\phi \to \psi$
(e) $\phi \infty \psi$ and $\phi \leftrightarrow \neg\psi$
(f) $\phi \infty \neg\psi$, $\neg\phi \infty \psi$, and $\phi \leftrightarrow \psi$

In a sense two equivalent formulas ϕ and ψ have the same meaning. We say that ϕ and ψ have the same *logical meaning*. So the remark made above can be given the following concise reformulation: logical meaning is conserved under replacement of a subformula by another formula which has the same logical meaning.

It is worth dwelling on the equivalence of $\phi \infty \psi$ and $\phi \leftrightarrow \neg\psi$ for a moment (exercise 7e). What this means is that *A unless B* and *A provided not B* have the same logical meaning: in logical terms then, (36) means the same as (37) (= (20)):

(36) We are going to see a film tonight, provided we are not going to the beach this afternoon.

(37) We are going to see a film tonight, unless we are going to the beach this afternoon.

Analogous points can be made with reference to the equivalences given in exercise 7f: *A unless not B* and *not A unless B* have the same logical meaning as *A provided B*, which means, among other things, that (38), (39) and (40) (= (29)) all express the same logical meaning:

(38) We are going to see a film tonight unless the dishes have not been done.

(39) We are not going to see a film tonight unless the dishes have been done.

(40) We are going to see a film tonight, provided the dishes have been done.

There are, of course, various reasons why one sentence may be preferred to another in any given context. What the equivalence of (38), (39), and (40) shows is that the reasons have nothing to do with the logical meaning of the sentences. The differences between these sentences are presumably to be explained in terms of their conditions of use, and it is there also that an explanation is to be sought for the peculiar nature of a sentence like:

(41) We are not going to see a film tonight provided we go to the beach this afternoon.

That there is a connection between material and logical equivalence is apparent if we compare the truth tables of the logically equivalent formulas p and $\neg\neg p$, and $p \wedge q$ and $q \wedge p$, with those of the material equivalences $p \leftrightarrow \neg\neg p$ and $(p \wedge q) \leftrightarrow (q \wedge p)$; see figures (42) and (43):

(42)

p	$\neg p$	$\neg\neg p$	$p \leftrightarrow \neg\neg p$
1	0	1	1
0	1	0	1

(43)

p	q	$p \wedge q$	$q \wedge p$	$(p \wedge q) \leftrightarrow (q \wedge p)$
1	1	1	1	1
1	0	0	0	1
0	1	0	0	1
0	0	0	0	1

In both cases we see that just one truth value occurs in the columns for the material equivalences, namely, 1. This is of course not entirely coincidental. It is precisely because under any valuation V, $V(p) = V(\neg\neg p)$ and $V(p \wedge q) = V(q \wedge p)$ that we always have $V(p \leftrightarrow \neg\neg p) = 1$ and $V((p \wedge q) \leftrightarrow (q \wedge p)) = 1$. Now this insight can be formulated as a general theorem:

Theorem 1

ϕ and ψ are logically equivalent iff for every valuation V, $V(\phi \leftrightarrow \psi) = 1$.

Proof: Generally speaking, a proof of a theorem of the form: A iff B is divided into (i) a proof that if A then B; and (ii) a proof that if B then A. The proof under (i) is headed by a $\Rightarrow$: and usually proceeds by first assuming A and then showing that B inevitably follows. The proof under (ii) is headed by a $\Leftarrow$: and usually proceeds by first assuming B and then showing that A inevitably follows. So the proof of our first theorem goes like this:

$\Rightarrow$: Suppose ϕ and ψ are logically equivalent. This means that for every valuation V for the propositional letters occurring in ϕ and ψ, $V(\phi) = V(\psi)$. Then condition (v) on valuations says that we must have $V(\phi \leftrightarrow \psi) = 1$.

$\Leftarrow$: Suppose that $V(\phi \leftrightarrow \psi) = 1$ for all valuations V. Then there can be no V such that $V(\phi) \neq V(\psi)$, since otherwise $V(\phi \leftrightarrow \psi) = 0$; so for every V it must hold that $V(\phi) = V(\psi)$, whence ϕ and ψ are logically equivalent. □

The box □ indicates that the proof has been completed.

In theorem 2 in §4.2.2 we shall see that formulas ϕ such that $V(\phi) = 1$ for every valuation V are of special interest. These formulas can be known to be true without any information concerning the truth of the parts of which they are composed. Such formulas ϕ are called *tautologies*, and that ϕ is a tautology is expressed by $\vDash \phi$. So theorem 1 can now be rewritten as follows:

$\vDash \phi \leftrightarrow \psi$ iff ϕ and ψ are logically equivalent.

Now theorem 1 gives us an ample supply of tautologies all at once, for example: $((\phi \vee \psi) \vee \chi) \leftrightarrow (\phi \vee (\psi \vee \chi))$, $(\phi \vee \psi) \leftrightarrow \neg(\neg\phi \wedge \neg\psi)$, de Morgan's laws, etc. And given that $\vDash \phi \rightarrow \psi$ and $\vDash \psi \rightarrow \phi$ whenever $\vDash \phi \leftrightarrow \psi$, we have even more. (This last is because if for every V, $V(\phi) = V(\psi)$, then we can be sure that for every V, $V(\phi) \leqslant V(\psi)$ and $V(\psi) \leqslant V(\phi)$.) As examples of tautologies we now have all formulas of the form $(\phi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\phi)$, and all those of the form $((\phi \vee \psi) \rightarrow \chi) \rightarrow ((\phi \rightarrow \chi) \wedge (\psi \rightarrow \chi))$. But there are many more, for example, all formulas of the form $\phi \rightarrow (\psi \rightarrow \phi)$, as is apparent from figure (44):

(44)

ϕ	ψ	$\psi \rightarrow \phi$	$\phi \rightarrow (\psi \rightarrow \phi)$
1	1	1	1
1	0	1	1
0	1	0	1
0	0	1	1

Exercise 8

Show of the following formulas that they are tautologies (for each ϕ, ψ, and χ):

(i) $\phi \rightarrow \phi$ (this actually follows from the equivalence of ϕ to itself)

(ii) $(\phi \wedge \psi) \rightarrow \phi$

(iii) $\phi \rightarrow (\phi \vee \psi)$

(iv) $\neg\phi \rightarrow (\phi \rightarrow \psi)$ (*ex falso sequitur quodlibet*)

(v) $\phi \vee \neg\phi$ (*law of the excluded middle*)

(vi) $(\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$

(vii) $(\phi \to \psi) \vee (\psi \to \phi)$

(viii) $((\phi \to \psi) \to \phi) \to \phi$ *(Peirce's law)*

Obviously all tautologies are equivalent to each other; if we always have $V(\phi) = 1$ and $V(\psi) = 1$, then we certainly always have $V(\phi) = V(\psi)$.

That a formula ϕ is not a tautology is expressed as $\not\models \phi$. If $\not\models \phi$, then there is a valuation V such that $V(\phi) = 0$. Any such V is called a *counterexample to* ϕ *('s being a tautology)*. In §4.2.1 we shall go into this terminology in more detail. As an example we take the formula $(p \to q) \to (\neg p \to \neg q)$, which can be considered as the schema for invalid arguments like this: *If one has money, then one has friends. So if one has no money, then one has no friends.* Consider the truth table in (45):

(45)

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \to \neg q$	$(p \to q) \to (\neg p \to \neg q)$
1	1	0	0	1	1	1
1	0	0	1	0	1	1
0	1	1	0	1	0	0
0	0	1	1	1	1	1

It appears that $\not\models (p \to q) \to (\neg p \to \neg q)$, since a 0 occurs in the third row of the truth table. This row is completely determined by the circumstance that $V(p) = 0$ and $V(q) = 1$, in the sense that for every valuation V with $V(p) = 0$ and $V(q) = 1$ we have $V((p \to q) \to (\neg p \to \neg q)) = 0$. For this reason we can say that $V(p) = 0$, $V(q) = 1$ is a counterexample to $(p \to q) \to (\neg p \to \neg q)$.

We must be very clear that in spite of this we cannot say whether a sentence of the form $(\phi \to \psi) \to (\neg \phi \to \neg \psi)$ is a tautology or not without more information about the ϕ and ψ. If, for example, we choose p for both ϕ and ψ, then we get the tautology $(p \to p) \to (\neg p \to \neg p)$, and if we choose $p \vee \neg p$ and q for ϕ and ψ, respectively, then we get the tautology $((p \vee \neg p) \to q) \to (\neg(p \vee \neg p) \to \neg q)$. But if we choose p and q for ϕ and ψ, respectively, then we arrive at the sentence $(p \to q) \to (\neg p \to \neg q)$, which, as we saw in (45), is not a tautology.

Exercise 9

Determine of the following formulas whether they are tautologies. If any is not, give a counterexample. (Why is this exercise formulated with p and q, and not with ϕ and ψ as in exercise 8?)

(i) $(p \to q) \to (q \to p)$

(ii) $p \vee (p \to q)$

(iii) $(\neg p \vee \neg q) \to \neg(p \vee q)$

(iv) $((p \vee q) \wedge (\neg p \to \neg q)) \to q$

(v) $((p \to q) \to p) \to ((p \to q) \to q)$

(vi) $((p \to q) \to r) \to (p \to (q \to r))$

Closely related to the tautologies are those sentences ϕ such that for every valuation V, $V(\phi) = 0$. Such formulas are called *contradictions*. Since they are never true, only to utter a contradiction is virtually to contradict oneself. Best known are those of the form $\phi \wedge \neg\phi$ (see figure (46)).

(46)

ϕ	$\neg\phi$	$\phi \wedge \neg\phi$
1	0	0
0	1	0

We can obtain many contradictions from

Theorem 2

If ϕ is a tautology, then $\neg\phi$ is a contradiction.

Proof: Suppose ϕ is a tautology. Then for every V, $V(\phi) = 1$. But then for every V it must hold that $(V \neg\phi) = 0$. So according to the definition, $\neg\phi$ is a contradiction. □

So $\neg((\phi \wedge \psi) \leftrightarrow (\psi \wedge \phi))$, $\neg(\phi \to \phi)$, and $\neg(\phi \vee \neg\phi)$ are contradictions, for example. An analogous proof gives us

Theorem 3

If ϕ is a contradiction, then $\neg\phi$ is a tautology.

This gives us some more tautologies of the form $\neg(\phi \wedge \neg\phi)$, the *law of noncontradiction*. All contradictions are equivalent, just like the tautologies. Those formulas which are neither tautologies nor contradictions are called *(logical) contingencies*. These are formulas ϕ such that there is both a valuation V_1 with $V_1(\phi) = 1$ and a valuation V_2 with $V_2(\phi) = 0$. The formula ϕ has, in other words, at least one 1 written under it in its truth table and at least one 0. Many formulas are contingent. Here are a few examples: p, q, $p \wedge q$, $p \to q$, $p \vee q$, etc. It should be clear that not all contingencies are equivalent to each other. One thing which can be said about them is:

Theorem 4

ϕ is a contingency iff $\neg\phi$ is a contingency.

Proof: (Another proof could be given from theorems 2 and 3, but this direct proof is no extra effort.)

$\Rightarrow$: Suppose ϕ is contingent. Then there is a V_1 with $V_1(\phi) = 1$ and a V_2 with $V_2(\phi) = 0$. But then we have $V_2(\neg\phi) = 1$ and $V_1(\neg\phi) = 0$, from which it appears that ϕ is contingent.

$\Leftarrow$: Proceeds just like $\Rightarrow$. □

Exercise 10

Let ϕ be a tautology, ψ a contradiction, and χ a contingency. Which of the following sentences are (i) tautological, (ii) contradictory, (iii) contingent, (iv) logically equivalent to χ.

(1) $\phi \wedge \chi$; (2) $\phi \vee \chi$; (3) $\psi \wedge \chi$; (4) $\psi \vee \chi$; (5) $\phi \wedge \psi$; (6) $\phi \vee \psi$; (7) $\chi \to \psi$.

Exercise 11

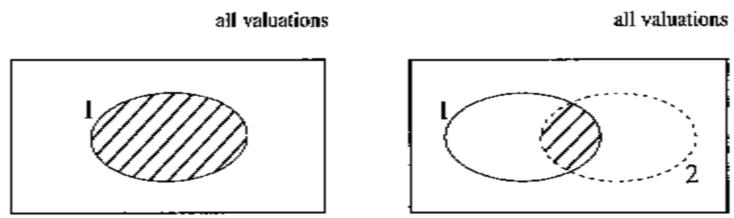
(i) Prove the following general assertions:
(a) If $\phi \rightarrow \psi$ is a contradiction, then ϕ is a tautology and ψ a contradiction.
(b) $\phi \wedge \psi$ is a tautology iff ϕ and ψ are both tautologies.

(ii) Refute the following general assertion by giving a formula to which it does not apply.
If $\phi \vee \psi$ is a tautology, then ϕ is a tautology or ψ is a tautology.

(iii) ◇ Prove the following general assertion:
If ϕ and ψ have no propositional letters in common, then $\phi \vee \psi$ is a tautology iff ϕ is a tautology or ψ is a tautology.

Before we give the wrong impression, we should emphasize that propositional logic is not just the science of tautologies or inference. Our semantics can just as well serve to model other important intellectual processes such as *accumulation of information*. Valuations on some set of propositional letters may be viewed as (descriptions of) states of the world, or situations, as far as they are expressible in this vocabulary. Every formula then restricts attention to those valuations ('worlds') where it holds: its 'information content'. More dynamically, successive new formulas in a discourse narrow down the possibilities, as in figure (47).

(47)



In the limiting case a unique description of one actual world may result. Note the inversion in the picture: the more worlds there still are in the information range, the less information it contains. Propositions can be viewed here as transformations on information contents, (in general) reducing uncertainty.

Exercise 12 ◇

Determine the valuations after the following three successive stages in a discourse (see (47)):

(1) $\neg(p \wedge (q \rightarrow r))$; (2) $\neg(p \wedge (q \rightarrow r))$, $(p \rightarrow r) \rightarrow r$; (3) $\neg(p \wedge (q \rightarrow r))$, $(p \rightarrow r) \rightarrow r$, $r \rightarrow (p \vee q)$.