5.5 Many-Valued Logic

5.5.1 *Introduction*

In standard propositional logic (and of course in standard predicate logic), formulas all end up with either 1 or 0 as their truth value. We say that classical logic is *two-valued*. In a two-valued logic, the formula known as the *principle of the excluded middle*, $\phi \vee \neg\phi$, is valid. But other systems with three or even an infinity of truth values have been developed for various reasons and for a variety of applications. Logical systems with more than two values are called many-valued logical systems, or many-valued logics.

In this section we will discuss several many-valued propositional logics, their intuitive bases, and their applications. Most attention will be paid to those aspects which are relevant to research into natural language. In particular, we will consider possible applications of many-valued logic in the analysis of the semantic concept of presupposition.

Many-valued propositional logics are not, in the sense introduced in §5.1, *extensions* of standard logic. They are what we have called *deviations* from standard propositional logic. Many-valued logical systems are not conceived in order to interpret more kinds of expressions but to rectify what is seen as a shortcoming in the existing interpretations of formulas. Once a new logical system has been developed, it often proves desirable and possible to introduce new kinds of expressions, and then the deviation becomes, in addition, an extension. But we shall begin with the familiar languages of standard propositional logic and show how a semantics with more than two truth values can be given for these.

5.5.2 *Three-Valued Logical Systems*

Since as far back as Aristotle, criticism of the principle of the excluded middle has been intimately linked to the status of propositions about contingent events in the future, and thus to the philosophical problem of determinism. This also applies to the three-valued system originated by the Polish logician Łukasiewicz, whose argument against bivalence derives from Aristotle's *sea battle argument*. Consider the sentence *A sea battle will be fought tomorrow*. This sentence states that a contingent event will take place in the future: it is possible that the sea battle will take place, but it is also possible that it will not. From this we can conclude that today the sentence is neither true nor false. For if the sentence were already true, then the sea battle would necessarily take place, and if it were already false, then it would be impossible for the sea battle to take place. Either way, this does not conform to the contingency of the sea battle. Accepting that propositions about future contingent events are now true or false amounts to accepting determinism and fatalism.

The validity of this argument is debatable. Its form can be represented as follows:

(56) $\phi \rightarrow$ necessary ϕ

(57) $\neg\phi \rightarrow$ impossible ϕ ($= \neg\phi \rightarrow$ necessary $\neg\phi$)

(58) $\phi \vee \neg\phi$

(59) necessary $\phi \vee$ necessary $\neg\phi$

In order to escape the deterministic conclusion (59), Aristotle rejected (58), the law of the excluded middle. These days, though, one would be much more inclined to think that something is wrong with premises (56) and (57) than with (58). From the truth of ϕ we cannot infer that *necessary* ϕ, and the same applies to falsity. From the falsity of ϕ we cannot conclude that *necessary* $\neg\phi$. In order to defend this conception properly, a logical analysis of the notion of necessity is required. One such analysis is given in modal logic, which is discussed in volume 2. There the (in)validity of arguments like the above is discussed in §2.3.5.

Although the original motivation for Łukasiewicz's many-valued logic is not watertight, it is interesting enough in its own right, since motivations other than the original one can be (and have been) given. Łukasiewicz's system can be given by means of the truth tables in (60):

(60)

ϕ	$\neg\phi$
1	0
#	#
0	1

$\phi \wedge \psi$

ϕ \ ψ	1	#	0
1	1	#	0
#	#	#	0
0	0	0	0

$\phi \vee \psi$

ϕ \ ψ	1	#	0
1	1	1	1
#	1	#	#
0	1	#	0

$\phi \rightarrow \psi$

ϕ \ ψ	1	#	0
1	1	#	0
#	1	1	#
0	1	1	1

The third value (#) stands for *indefinite* or *possible*. It should be clear how these tables should be read. They are slightly different in form from the truth tables we have dealt with so far. Figure (61a) shows how the two-valued truth table for the conjunction can be written in this manner. And figure (61b) shows how the three-valued conjunction can be written in the original way.

(61) a.

$\phi \wedge \psi$

ϕ \ ψ	1	0
1	1	0
0	0	0

b.

ϕ	ψ	$\phi \wedge \psi$
1	1	1
1	#	#
1	0	0
#	1	#
#	#	#
#	0	0
0	1	0
0	#	0
0	0	0

Tables like those in (61) are useful if we only want to say how the connectives should be interpreted, but we have to stick with the original way of writing truth tables if we want to use them for calculating the truth values of composite formulas from the truth values of the proposition letters in them.

According to the table for negation in (60), the value of ϕ is always indeterminate if the value of $\neg\phi$ is. And from the table for disjunction it follows that *the law of the excluded middle* does not hold. As can be seen from (62), $\phi \vee \neg\phi$ never has the truth value 0, but it doesn't always have the value 1 either. If ϕ has # as its truth value, then $\neg\phi$ has value # too.

(62)

ϕ	$\neg\phi$	$\phi \vee \neg\phi$
1	0	1
#	#	#
0	1	1

It follows similarly from the table for conjunction that *the law of noncontradiction* $\neg(\phi \wedge \neg\phi)$ does not hold. The *law of identity*, on the other hand, does hold: $\phi \rightarrow \phi$ is valid, since according to (63) it always has 1 as its truth value.

(63)

ϕ	$\phi \rightarrow \phi$
1	1
#	1
0	1

This is because according to the table for implication in (60), if ϕ has # as its truth value, then $\phi \rightarrow \phi$ has, not #, but 1 as its truth value. Related to this is the fact that while the interdefinability of $\vee$ and $\wedge$ by means of $\neg$ still holds, the interdefinability of $\vee$ and $\rightarrow$, or that of $\wedge$ and $\rightarrow$, does not. The reason for this is that both $\phi \vee \psi$ and $\phi \wedge \psi$ have truth value # if both ϕ and ψ have value #, while $\phi \rightarrow \psi$ has 1 as its truth value in that case.

Kleene has proposed a three-valued system which differs from Łukasiewicz's on exactly this point. His interpretation for $\rightarrow$ is given in (64):

(64) a.

ϕ	$\neg\phi$
1	0
#	#
0	1

b. $\phi \wedge \psi$

ϕ \ ψ	1	#	0
1	1	#	0
#	#	#	0
0	0	0	0

c. $\phi \vee \psi$

ϕ \ ψ	1	#	0
1	1	1	1
#	1	#	#
0	1	#	0

d. $\phi \rightarrow \psi$

ϕ \ ψ	1	#	0
1	1	#	0
#	1	#	#
0	1	1	1

Although Kleene's system only differs from Łukasiewicz's system in the implication, we have listed it completely in (64), since we will often want to refer to it in what is to come. According to Kleene's table for implication, $\phi \rightarrow \phi$ is no longer a valid formula. On the other hand, the interdefinability of $\vee$ and $\rightarrow$ via negation, as well as that of $\wedge$ and $\rightarrow$, has been regained. Kleene interprets the third value not as 'indefinite' but as 'undefined'. The value of a composite formula can be definite or defined even if the value of one or more of its parts is not. This is the case if the known value of some part is enough to

decide the value of the whole formula. We know, for example, that $\phi \rightarrow \psi$ is always true if its antecedent is false, whatever the value of the consequent is. So if ϕ has the value 0, then $\phi \rightarrow \psi$ has the value 1, whether or not ψ is #.

One undesirable result of the interpretation of the third value as 'undefined' is that the truth value of $\phi \vee \neg\phi$ is undefined if that of ϕ is. This is not very satisfactory, since even if it is not known yet what the value of ϕ is, it certainly is clear that the value of ϕ depends on that of $\neg\phi$. We don't know what value ϕ has, but we do know that $\neg\phi$ has the value 1 if ϕ has value 0, and vice versa. So one could argue that we know that $\phi \vee \neg\phi$ has 1 as its truth value, even if we do not know yet what truth value ϕ has.

Van Fraassen's method of *supervaluations* was developed in order to meet this difficulty. This method gives all formulas which have the same value under all valuations in standard logic (that is, the tautologies and contradictions of standard logic) that same value. But the same does not apply to contingent formulas. We shall not further discuss supervaluations here.

Another sort of three-valued system originates when the third value is interpreted as *meaningless* or *nonsense*, and Bochvar proposed the three-valued system presented in (65) with this interpretation in mind

(65) a.

ϕ	$\neg\phi$
1	0
#	#
0	1

b. $\phi \wedge \psi$

ϕ \ ψ	1	#	0
1	1	#	0
#	#	#	#
0	0	#	0

c. $\phi \vee \psi$

ϕ \ ψ	1	#	0
1	1	#	1
#	#	#	#
0	1	#	0

d. $\phi \rightarrow \psi$

ϕ \ ψ	1	#	0
1	1	#	0
#	#	#	#
0	1	#	1

The third value in (65) is dominant in the sense that a composite formula receives # as its value whenever any of its composite parts does. If any part of a sentence is nonsense, then the sentence as a whole is nonsense. This interpretation of the connectives is known as the *weak interpretation*, this in contradistinction to Kleene's *strong interpretation*. Łukasiewicz's, Kleene's, and Bochvar's systems all agree in giving the same truth value as classical logic to any formula whose subformulas all have classical truth values. Bochvar's system differs from the other two in that if a formula has a classical truth value in his system, then all of its subformulas must too. As we have just seen, in Łukasiewicz's and Kleene's system, a formula can have a classical truth value even if some of its subformulas do not.

5.5.3 *Three-Valued Logics and the Semantic Notion of Presupposition*

One important if much-debated application of three-valued logic in linguistics is in dealing with *presupposition*. In §5.2 we saw how Russell's theory of descriptions analyzes sentences with definite descriptions, like (66) and (67):

(66) The king of France is bald.

(67) The queen of the Netherlands is riding a bicycle.

His theory analyzes the sentences in such a way that the existence of a king of France and a queen of the Netherlands are among the things which these sentences state. A sentence like (66) is then, according to Russell, false. Russell's analysis of definite descriptions was criticized by Strawson in 'On Referring' (1950). According to Strawson, Russell's theory gives a distorted picture of the way definite descriptions are used. That there is a king of France is not something which is being stated when sentence (66) is stated; it is something which is assumed by (66), a *presupposition*. And if there is no king of France, then sentence (66) is not false, since then there is no proposition of which it can be said that it is true or false.

It has always been a moot point what field the concept of presupposition belongs to, semantics or pragmatics. If it belongs to semantics, then the falsity of a presupposition affects the truth value of a sentence. And if it belongs to pragmatics, then the concept of presupposition must be described in terms of the ways we use language. In order to utter a sentence correctly, a speaker, for example, must believe all of its presuppositions. We shall not attempt to decide the issue here. But in chapter 6 we return to the distinction between semantic and pragmatic aspects of meaning.

In the following, we shall restrict ourselves in the examples to the *existential presuppositions* of definite descriptions. The *existential presupposition* of a definite description is the assumption that there is some individual answering to it. There is also the *presupposition of uniqueness*, which is the presupposition that no more than a single individual answers to it. And other kinds of expressions have their own special kinds of presuppositions. Verbs and verb phrases like *to know* and *to be furious* have *factive presuppositions*, for example. Sentences (68) and (69) both presuppose that John kissed Mary:

(68) Peter knows that John kissed Mary.

(69) Peter is furious that John kissed Mary.

Verbs like *to believe* and *to say*, on the other hand, do not carry factive presuppositions. One last example.

(70) All of John's children are bald.

A sentence like (70) also has an existential presupposition, namely, that John has children.

Proponents of many-valued logic in the analysis of presupposition see it as a *semantic* concept. Strawson's position is then presented as follows. If one of a sentence's presuppositions is not true, then the sentence is neither true nor false, but has a third truth value. A mistaken presupposition would thus affect the truth value of a sentence. This approach leads to the following definition for presuppositions:

Definition 3

ψ is a *presupposition* of ϕ iff for all valuations V: if $V(\psi) \neq 1$, then $V(\phi) \neq 1$ and $V(\phi) \neq 0$.

In a three-valued system, this means that if $V(\phi) \neq 1$ and $V(\phi) \neq 0$, then $V(\phi) = \#$. So definition 3 is equivalent to the more usual formulation:

(71) ψ is a presupposition of ϕ iff for all valuations V: if $V(\psi) \neq 1$, then $V(\phi) = \#$.

Negation has been the same in all three-valued systems we have seen so far. In particular, in all cases, $V(\phi) = \#$ iff $V(\neg\phi) = \#$. Together with (71), this gives us (72):

(72) ψ is a presupposition of ϕ just in case ψ is a presupposition of $\neg\phi$.

This property is considered characteristic of presuppositions. Not only sentences (66) and (67) but also their negations, (73) and (74), respectively, presuppose the existence of a French king and a Dutch queen:

(73) The king of France is not bald.

(74) The queen of the Netherlands is not riding a bicycle.

We could also have taken the fact that presuppositions are retained under negation as our starting point and used it as an argument in favor of many-valued logic in the analysis of semantic presuppositions. We could then reason as follows: Both the truth of (67) and that of its negation (74) 'imply' the truth of (75):

(75) There is a queen of the Netherlands.

But then the implicational relation between (67) and (75) and that between (74) and (75) cannot be a normal notion of logical inference in a two-valued system, since in any such logical system, tautologies are the only formulas implied by both a formula and its negation, while (75) is clearly a contingent proposition. This can be seen as follows. That both ϕ and $\neg\phi$ 'imply' the formula ψ means:

(76) For all valuations V: if $V(\phi) = 1$, then $V(\psi) = 1$; and if $V(\neg\phi) = 1$, then $V(\psi) = 1$.

This is equivalent to:

(77) For all valuations V: if $V(\phi) = 1$ or $V(\neg\phi) = 1$, then $V(\psi) = 1$.

But the antecedent of (77), $V(\phi) = 1$ or $V(\neg\phi) = 1$, is always true in a two-valued system, so (77) amounts to:

(78) For all valuations V: $V(\psi) = 1$.

That is, ψ is a tautology. The remedy is to abandon bivalence, that is, the requirement that for any sentence ϕ, either $V(\phi) = 1$ or $V(\neg\phi) = 1$. (Russell had a different remedy: withdraw the assumption that (74) is the direct negation of (67).) In a three-valued system with $\neg$ defined as in the tables in §5.5.2, (77) is equivalent to the definiendum of definition 3, the definition of presupposition. So a preference for a semantic treatment of the concept of presupposition gives us an argument in favor of three-valued logic.

In §5.2.2 we presented a number of different three-valued systems. The question arises at this point as to which of these systems is best suited for a treatment of presupposition. This question is related to the way the presuppositions of a composite sentence depend on the presuppositions of its composite parts, which is known as the *projection problem for presuppositions.* As we shall see, the different many-valued systems with their different truth tables for the connectives give different answers.

If we choose Bochvar's system, in which composite sentences receive # as their value whenever any of their composite parts does, presupposition becomes *cumulative.* The presuppositions of a composite sentence are just all the presuppositions of its composite parts. If any presupposition of any of the composite parts fails, then a presupposition of the sentence as a whole likewise fails. If a presupposition of any of the composite parts does not have 1 as its truth value, then the whole formula has # as its truth value. This follows directly from the truth tables given for the connectives in (65) and definition 3.

If we add a new operator P to our propositional languages, then $P\phi$ can stand for the presuppositions of ϕ. We define this operator as in (79):

(79)

ϕ	$P\phi$
1	1
#	0
0	1

The formula $P\phi$ is equivalent to the necessary and sufficient conditions for the satisfaction of ϕ's presuppositions. The formula $P\phi$ receives value 1 if all ϕ's presuppositions are satisfied, and otherwise it receives 0. $P\phi$ itself does not have any presupposition, since it never receives # as its value. $PP\phi$ is always a tautology. The logical consequences of $P\phi$ are precisely the presuppositions of ϕ. It can easily be checked that the following equivalences hold by constructing truth tables:

(80) $P\phi$ and $P\neg\phi$ are equivalent.

(81) $P(\phi \vee \psi)$, $P(\phi \wedge \psi)$, and $P(\phi \rightarrow \psi)$ are equivalent to $P\phi \wedge P\psi$.

Here (80) is just a reformulation of the characteristic property of presupposition already given as (72), namely, that ϕ and $\neg\phi$ have the same presuppositions. What (81) says is that if presuppositions are cumulative, then the presuppositions of a conjunction and a disjunction can be written as the conjunction of the presuppositions of its conjuncts and disjuncts, respectively, and the presupposition of an implication can be written as the conjunction of the presupposition of its antecedent and its consequent. This is because # appears in the same places in the truth tables of all three connectives in Bochvar's system. The value # appears whenever any of the formulas joined by the connective has # as its value. (See (65).)

So by using Bochvar's system, we obtain a cumulative notion of presupposition. But presupposition is generally thought not to be cumulative. There are cases in which presuppositions are, as we say, *canceled* in the formation of composite formulas, and this makes the projection problem much more interesting. Sentences (82)–(84) are clear examples of the fact that a formula does not need to inherit all the presuppositions of its subformulas:

(82) If there is a king of France, then the king of France is bald.

(83) Either there is no king of France or the king of France is bald.

(84) There is a king of France and the king of France is bald.

Sentence (85):

(85) The king of France is bald.

is a part of (82), (83), and (84). Sentence (86):

(86) There is a king of France.

is a presupposition of (85), but not of (82)–(84). If sentence (86) is false, then (82) and (83) are true, and (84) is false. This can be explained if we choose, not Bochvar's system, but Kleene's. A sentence like (82) is of the form $p \rightarrow q$, in which p is a presupposition of q. That p is not a presupposition of $p \rightarrow q$ in Kleene's system can now be seen as follows. Suppose p has value 0; then q has #, since p is one of its presuppositions. But according to Kleene's truth table for implication, the whole implication still has 1 as its value, since its antecedent has value 0. So according to definition 3, p is not a presupposition of $p \rightarrow q$, since although in this case, p does not have 1 as its value, $p \rightarrow q$ still doesn't have # as its value. Something similar holds for sentence (83), which is of the form $\neg p \vee q$, in which p is once again a presupposition of q. If p has value 0 (in which case q has #), then Kleene's table for $\vee$ still results in $\neg p \vee q$ having 1 as its truth value. Sentence (84), finally, has the form $p \wedge q$, with p again a presupposition of q. Now if p has truth value 0, then so does the whole conjunction, in spite of the fact that q has # as its truth value. So Kleene's three-valued system explains why (86), a presupposition of formula (85), is canceled when the latter is incorporated into composite sentences (82)–(84). In §5.5.6 we will see, however, that Kleene's system is not the last word in the analysis of presuppositions.

As in Bochvar's system, ϕ's presuppositions can be represented by means of $P\phi$ in Kleene's system. Since negation is the same in both, equivalence (80) still holds. But since the other connectives are different, the equivalences in (81) no longer hold. Instead we have the somewhat more complicated equivalences in (87)–(89).

(87) $P(\phi \vee \psi)$ is equivalent to $((\phi \wedge P\phi) \vee P\psi) \wedge ((\psi \wedge P\psi) \vee P\phi)$.

(88) $P(\phi \wedge \psi)$ is equivalent to $((\neg\phi \wedge P\phi) \vee P\psi) \wedge ((\neg\psi \wedge P\psi) \vee P\phi)$.

(89) $P(\phi \rightarrow \psi)$ is equivalent to $((\neg\phi \wedge P\phi) \vee P\psi) \wedge ((\psi \wedge P\psi) \vee P\phi)$.

We now introduce a second operator A to be interpreted according to (90):

(90)

ϕ	$A\phi$
1	1
#	0
0	0

Then as a result of the equivalence of $A\phi$ and $\phi \wedge P\phi$ (87)–(89) amount to (91)–(93):

(91) $P(\phi \vee \psi)$ is equivalent to $(A\phi \vee P\psi) \wedge (A\psi \vee P\phi)$.

(92) $P(\phi \wedge \psi)$ is equivalent to $(A\neg\phi \vee P\psi) \wedge (A\neg\psi \vee P\phi)$.

(93) $P(\phi \rightarrow \psi)$ is equivalent to $(A\neg\phi \vee P\psi) \wedge (A\psi \vee P\phi)$.

A third way of writing $P(\phi \vee \psi)$, which avoids A, is:

(94) $P(\phi \vee \psi)$ is equivalent to $(\phi \vee P\psi) \wedge (\psi \vee P\phi) \wedge (P\phi \vee P\psi)$.

Equations resembling (94) can of course also be given for the other two connectives.

The P-operator can also be used to clarify the cancellation of presuppositions in Kleene's system. If (86) is the only presupposition of (85) then writing q for (85) (and thus Pq for (86)), sentences (82)–(84) can be represented as follows:

(95) $Pq \rightarrow q$

(96) $\neg Pq \vee q$

(97) $Pq \wedge q$

That q's presupposition Pq is canceled in the formation of (95)–(97) is apparent from the fact that (95)–(97) themselves have no presuppositions at all, or more precisely, that they only have tautologies as their presuppositions. The formulas $P(Pq \rightarrow q)$, $P(\neg Pq \vee q)$, and $P(Pq \wedge q)$ are tautologies; they always have 1 as their truth value. This explains why the contingent sentence (86) is not a presupposition of (82)–(84).

Equivalences like those in (87)–(89) and (91)–(94) are interesting on more than one account. First, they shed some light on how the projection problem for presupposition is approached in a three-valued system like Kleene's. For example, (87) says directly that the presuppositions of $(\phi \vee \psi)$ are satisfied in each of the following three cases: if the presuppositions of both ϕ and ψ are satisfied (compare this with cumulative presupposition); if ϕ's presuppositions are not satisfied, but ψ is true; and last, if ψ's presuppositions are not satisfied, but ϕ is true. That this concept of presupposition is weaker than the cumulative one is because of the last two cases. They correspond to the two places in Kleene's table for $\vee$ (see (64)) in which there is a 1 instead of the # in Bochvar's system (see (65)).

A second reason why these equivalences are interesting is that they have much in common with the inductive definitions of the concept of a presupposition which have been published as an alternative to three-valued approaches. These definitions inductively define a formula ϕ^{Pr} which amounts to the set of ϕ's presuppositions. They begin by stipulating what the presuppositions of atomic formulas are. The inductive clauses are then, for example:

(98) $(\neg\phi)^{Pr} = \phi^{Pr}$

(99) $(\phi \vee \psi)^{Pr} = ((\phi \wedge \phi^{Pr}) \vee \psi^{Pr}) \wedge ((\psi \wedge \psi^{Pr}) \vee \phi^{Pr})$

The remaining connectives have something similar. In the literature it is common to speak in terms of the set of a formula's presuppositions. The approach sketched here amounts to forming the conjunction of all formulas in such a set. It has been suggested that this kind of inductive definition is more adequate than a treatment in terms of a three-valued semantics. But in view of the similarity of (87) and (99) it seems likely that both approaches give the same results.

Although a three-valued system like Kleene's deals satisfactorily with certain aspects of the projection problem for presuppositions, it leaves certain problems open. These will be discussed to some extent in §5.5.6. But first we shall describe many-valued logical systems with more than three values (§5.5.4) and their applications in the analysis of the semantic notion of presupposition (§5.5.5).

5.5.4 *Logical systems with more than three values*

So far the discussion of many-valued logical systems has gone no further than three-valued systems. But logics with more than three values have also been developed. A system like Kleene's, for example, can easily be generalized to systems with any finite number n ($n \geq 2$) of truth values. One convenient notation for the truth values of such a system uses fractions, with the number $n - 1$ as their denominator and the numbers $0, 1, \ldots, n - 1$ as their numerators. The three-valued system ($n = 3$) then has the truth values $\frac{0}{2}$, $\frac{1}{2}$, and $\frac{2}{2}$, or 0, $\frac{1}{2}$, and 1. So the third value of the Kleene system is written as $\frac{1}{2}$ instead of as #. A four-valued system ($n = 4$) then has the truth values $\frac{0}{3}$, $\frac{1}{3}$, $\frac{2}{3}$, and $\frac{3}{3}$, or 0, $\frac{1}{3}$, $\frac{2}{3}$, and 1. The truth values of composite formulas in a Kleene system with n truth values can now be calculated as follows:

Definition 4

$V(\neg\phi)$	$= 1 - V(\phi)$
$V(\phi \wedge \psi)$	$= V(\phi)$ if $V(\phi) \leq V(\psi)$
	$= V(\psi)$ otherwise
$V(\phi \vee \psi)$	$= V(\phi)$ if $V(\phi) \geq V(\psi)$
	$= V(\psi)$ otherwise
$V(\phi \to \psi)$	$= V(\psi)$ if $V(\psi) \geq (1 - V(\phi))$
	$= 1 - V(\phi)$ otherwise

So a conjunction is given the truth value of whichever of its conjuncts has the lowest truth value; a disjunction is given the truth value of whichever of its disjuncts has the highest truth value. The truth value of the implication $\phi \to \psi$ is equal to that of the disjunction $\neg\phi \vee \psi$. For a three-valued system, the truth tables are the same as those in (64), but with $\frac{1}{2}$ instead of #. For $n = 2$ this reduces to standard propositional logic. A four-valued Kleene system has the truth tables given in (100):

(100)

ϕ	$\neg\phi$
1	0
$\frac{2}{3}$	$\frac{1}{3}$
$\frac{1}{3}$	$\frac{2}{3}$
0	1

$\phi \wedge \psi$

ϕ \ ψ	1	$\frac{2}{3}$	$\frac{1}{3}$	0
1	1	$\frac{2}{3}$	$\frac{1}{3}$	0
$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	0
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
0	0	0	0	0

$\phi \vee \psi$

ϕ \ ψ	1	$\frac{2}{3}$	$\frac{1}{3}$	0
1	1	1	1	1
$\frac{2}{3}$	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$\frac{1}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
0	1	$\frac{2}{3}$	$\frac{1}{3}$	0

$\phi \to \psi$

ϕ \ ψ	1	$\frac{2}{3}$	$\frac{1}{3}$	0
1	1	$\frac{2}{3}$	$\frac{1}{3}$	0
$\frac{2}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\frac{1}{3}$	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
0	1	1	1	1

Similarly, a Kleene system can have infinitely many truth values, for example, by taking all fractions between 0 and 1 as truth values.

The above systems with more than three values are all obtained by generalizing a system with three values. Other systems with more than three values can be obtained, for example, by 'multiplying' systems by each other. These are called *product systems*. In such a product of two systems S_1 and S_2, formulas are given truth values $\langle v_1, v_2\rangle$, in which v_1 derives from S_1, and v_2 from S_2. A product system can be applied if we want to evaluate formulas under two different and independent aspects and to represent the evaluations in combination. We can, for example, multiply the standard two-valued system by itself. We then obtain a four-valued system with the pairs $\langle 1, 1\rangle$, $\langle 1, 0\rangle$, $\langle 0, 1\rangle$, and $\langle 0, 0\rangle$ as its truth values. In order to calculate the truth value of a formula in the product system, we must first calculate its truth value in each of the two systems of which it is a product. The value in the first system becomes the first member of the ordered pair, and the value in the second system is the second member. The truth tables for the connectives for this four-valued system are as in (101) (we write 11 instead of $\langle 1, 1\rangle$, etc.).

(101)

ϕ	$\neg\phi$
11	00
10	01
01	10
00	11

$\phi \wedge \psi$

ϕ \ ψ	11	10	01	00
11	11	10	01	00
10	10	10	00	00
01	01	00	01	00
00	00	00	00	00

$\phi \vee \psi$

ϕ \ ψ	11	10	01	00
11	11	11	11	11
10	11	10	11	10
01	11	11	01	01
00	11	10	01	00

$\phi \to \psi$

ϕ \ ψ	11	10	01	00
11	11	10	01	00
10	11	11	01	01
01	11	10	11	10
00	11	11	11	11

Systems with different kinds and numbers of truth values can, of course, just as easily be multiplied by each other. If one system has m truth values and the other n, then the product system will have $m \times n$ truth values.

that here.

5.5.6 *The Limits of Many-Valued Logics in the Analysis of Presupposition*

There are a few ways that a three- or four-valued Kleene system can furnish a good explanation for the cancellation of presuppositions in the composition of sentences. But there are also certain problems. The first problem is displayed very clearly in sentences like (113) and (114).

(113) The king of France is not bald, since there is no king of France.

(114) There is no king of France; thus the king of France is not bald.

If (113) is true, then both (115) and (116) must be true (and the same applies for (114)):

(115) The king of France is not bald.

(116) There is no king of France.

But the problem is that (115) and (116) cannot both be true at the same time, because (116) is the negation of one of the presuppositions of (115). What we seem to need here is for (115) to be true even if there isn't any king of France. This does not present any problem for Russell's theory of descriptions, since according to Russell, sentence (115) is ambiguous (see §5.2). A similar solution can be found within a many-valued system. We distinguish two readings of (115), introducing for this purpose a new kind of negation, $\sim$. This negation is defined by the table in (117):

(117)

ϕ	$\sim\phi$
1	0
#	1
0	1

If p is a presupposition of q, then $\sim$q is true according to the table for $\sim$ if p is not true. The negation $\sim$ is called *internal negation* and $\neg$ *external negation*. Interpreting the negation in (115) as external negation, (115) and (116) can both be true at the same time, so (113) and (114) can both be true.

Note that the operators A and $\sim$ are interdefinable via $\neg$: $\sim\phi$ is equivalent to $\neg A\phi$ (and thus $A\phi$ is equivalent to $\neg\sim\phi$). By introducing operators like $\sim$, A, and P we have extended standard propositional logic by adding new logical constants. But the introduction of these operators only makes sense if we choose a many-valued interpretation. This is what we meant in §5.5.1 when we said that deviations from standard interpretations often give rise to extensions.

The problem with sentences like (113) and (114) can then be solved by distinguishing two different kinds of negation. This seems a bit ad hoc, since we have not given any systematic way to determine whether a negation should be given the internal or the external reading. There is, however, an even more serious problem than that with (113) and (114). It can be illustrated by means of a sentence like:

(118) If baldness is hereditary, then the king of France is bald.

Now intuitively it is clear enough that one presupposition of (118) is (119) (= (86)):

(119) There is a king of France.

According to definition 3, then, no valuation which renders (119) false may render (118) true. But consider (120):

(120) Baldness is hereditary.

Even though (120) is the antecedent of (118), it can be false without there being a king of France, since sentences (119) and (120) are logically independent of each other. So let V be any valuation which renders both sentences false. Then V renders (118) true, since it renders its antecedent (120) false. So

V renders (119) false and (118) true, in contradiction with the above remark that (119) is a presupposition of (118). In general the problem is this: implications with contingent antecedents which are logically independent of certain presuppositions of their consequents have too many of their presuppositions canceled. Similar complications arise with the other connectives.

At this point various lines of action could be taken. One idea would be to try to find better many-valued definitions for the connectives. Bochvar's system would do very well for sentences like (118), but it has its own problems with sentences like (82)–(84). So far no single system has been found which deals with both (82)–(84) and (118) in a satisfactory manner, and it is questionable if there is any such system to be found. A second possibility might be to adapt definition 3. This has been tried a couple of times, and the results have not been satisfactory.

A third idea, which has so far been the most successful, is to stick to both a three- or four-valued Kleene system and definition 3, but to drop the idea that the presupposition of (118) that there is a king of France is a *semantic* presupposition. We then give a *pragmatic* explanation for the fact that anyone who asserts (118) or its negation must believe that France has a king. This means that we must introduce the notion of *pragmatic presuppositions* as a complement to the semantic notion. Any such pragmatic explanation must lean heavily on Grice's theory of *conversational maxims*. We shall return to these maxims at length in chapter 6, where we shall also briefly consider the possibility that presupposition is not a semantic notion at all but must be wholly explained in pragmatic terms.