3.7 Identity

It is often useful in languages for predicate logic to have a binary predicate letter which expresses *identity*, the *equality* of two things. For this reason, we now introduce a new logical constant, $=$, which will always be interpreted as the relation of identity. The symbol $=$, of course, has been used many times in this book as an informal equality symbol derived from natural language or, if the reader prefers, as a symbol which is commonly added to natural language in order to express equality. We will continue to use $=$ in this informal way, but this need not lead to any confusion.

A strong sense of the notion *identity* is intended here: by $a = b$ we do not mean that the entities to which a and b refer are identical in the sense that they resemble each other very closely, like identical twins, for example. What we mean is that they are the same, so that $a = b$ is true just in case a and b refer to the same entity. To put this in terms of valuations, we want $V_M(a = b) = 1$ in any model **M** just in case $I(a) = I(b)$. (The first $=$ in the sentence was in a formal language, the object language; the other two were in natural language, the metalanguage.)

The right valuations can be obtained if we stipulate that I will always be such that: $I(=) = \{\langle d, e\rangle \in D^2 \mid d = e\}$, or a shorter notation: $I(=) = \{\langle d, d\rangle \mid d \in D\}$. Then, with approach A, we have $V_M(a = b) = 1$ iff $\langle I(a), I(b)\rangle \in I(=)$ iff $I(a) = I(b)$. And with method B, we have $V_{M,g}(a = b) = 1$ iff $\langle [\![a]\!]_{M,g}, [\![b]\!]_{M,g}\rangle \in I(=)$ iff $[\![a]\!]_{M,g} = [\![b]\!]_{M,g}$ iff $I(a) = I(b)$.

The identity symbol can be used for more than just translations of sentences like *The morning star is the evening star* and *Shakespeare and Bacon are one and the same person*. Some have been given in (98):

(98)

Sentence	*Translation*
John loves Mary, but Mary loves someone else.	$Ljm \wedge \exists x(Lmx \wedge x \neq j)$
John does not love Mary but someone else.	$\neg Ljm \wedge \exists x(Ljx \wedge x \neq m)$

John loves no one but Mary.	$\forall x(Ljx \leftrightarrow x = m)$
No one but John loves Mary.	$\forall x(Lxm \leftrightarrow x = j)$
John loves everyone except Mary.	$\forall x(Ljx \leftrightarrow x \neq m)$
Everyone loves Mary except John.	$\forall x(Lxm \leftrightarrow x \neq j)$

The keys to the translations are the obvious ones and have been left out. In all cases, the domain is one with just people in it. We shall always write $s \neq t$ instead of $\neg(s = t)$.

If the domain in the above examples were to include things other than people, then $\forall x(Hx \rightarrow$ would have to be substituted for $\forall x$ in all the translations, and $\exists x(Hx \wedge$ for $\exists x$. Quite generally, if a sentence says that of all entities which have some property A, only a bears the relation R to b, then that sentence can be translated as $\forall x(Ax \rightarrow (Rxb \leftrightarrow x = a))$; but if a sentence states that all entities which have A bear R to b except the one entity a, then that sentence can be translated as $\forall x(Ax \rightarrow (Rxb \leftrightarrow x \neq a))$. We can also handle more complicated sentences, such as (99):

(99) Only John loves no one but Mary.

Sentence (99) can be rendered as $\forall x(\forall y(Lxy \leftrightarrow y = m) \leftrightarrow x = j)$. That this is correct should be fairly clear if it is remembered that $\forall y(Lxy \leftrightarrow y = m)$ says that x loves no one but Mary.

One of Frege's discoveries was that the meanings of numerals can be expressed by means of the quantifiers of predicate logic and identity. The principle behind this is illustrated in (100), the last three rows of which contain sentences expressing the numerals *one, two,* and *three*. For any natural number *n*, we can express the proposition that there are *at least n* things which have some property A by saying that there are *n* mutually different things which have A. That there are *at most n* different things which have A can be expressed by saying that of any $n + 1$ (not necessarily different) things which have A, at least two must be identical. That there are *exactly n* entities with A can now be expressed by saying that there are at least, and at most, *n* entities with A. So, for example, $\exists xAx \wedge \forall x\forall y((Ax \wedge Ay) \rightarrow x = y)$ can be used to say that there is exactly one x such that Ax. But shorter formulas that have the same effect can be found if we follow the procedure illustrated in (100). We say that there are *n* different entities and that any entity which has the property A must be one of these.

(100)

There is at least one x such that Ax.	$\exists xAx$
There are at least two (different) x such that Ax.	$\exists x\exists y(x \neq y \wedge Ax \wedge Ay)$
There are at least three (different) x such that Ax.	$\exists x\exists y\exists z(x \neq y \wedge x \neq z \wedge y \neq z \wedge Ax \wedge Ay \wedge Az)$
There is at most one x such that Ax.	$\forall x\forall y((Ax \wedge Ay) \rightarrow x = y)$
There are at most two (different) x such that Ax.	$\forall x\forall y\forall z((Ax \wedge Ay \wedge Az) \rightarrow (x = y \vee x = z \vee y = z))$
There are at most three (different) x such that Ax.	$\forall x\forall y\forall z\forall w((Ax \wedge Ay \wedge Az \wedge Aw) \rightarrow (x = y \vee x = z \vee x = w \vee y = z \vee y = w \vee z = w))$
There is exactly one x such that Ax.	$\exists x\forall y(Ay \leftrightarrow y = x)$
There are exactly two x such that Ax.	$\exists x\exists y(x \neq y \wedge \forall z(Az \leftrightarrow (z = x \vee z = y)))$
There are exactly three x such that Ax.	$\exists x\exists y\exists z(x \neq y \wedge x \neq z \wedge y \neq z \wedge \forall w(Aw \leftrightarrow (w = x \vee w = y \vee w = z)))$

This procedure is illustrated for a unary predicate letter A, but it works just as well for formulas ϕ. The formula $\exists x\forall y([y/x]\phi \leftrightarrow y = x)$, for example, says that there is exactly one thing such that ϕ, with the proviso that y must be a variable which is free for x in ϕ and does not occur free in ϕ. Sometimes a special notation is used for a sentence expressing *There is exactly one x such that* ϕ, $\exists x\forall y([y/x]\phi \leftrightarrow y = x)$ being abbreviated as $\exists!x\phi$.

We now give a few examples of sentences which can be translated by means of $=$. We do not specify the domains, since any set which is large enough will do.

(101) There is just one queen.
Translation: $\exists x\forall y\ (Qy \leftrightarrow y = x)$.
Key: Qx: x is a queen.

(102) There is just one queen, who is the head of state.
Translation: $\exists x(\forall y(Qy \leftrightarrow y = x) \wedge x = h)$.
Key: Qx: x is a queen; h: the head of state.
(This should be contrasted with $\exists!x(Qx \wedge x = h)$, which expresses that only one person is a governing queen, although there may be other queens around.)

(103) Two toddlers are sitting on a fence.
Translation: $\exists x(Fx \wedge \exists y_1\exists y_2(y_1 \neq y_2 \wedge \forall z((Tz \wedge Szx) \leftrightarrow$

$(z = y_1 \vee z = y_2))))$.
Key: Tx: x is a toddler; Sxy: x is sitting on y; Fx: x is a fence.

(104) If two people fight for something, another will win it.

Translation: $\forall x \forall y \forall z((Px \wedge Py \wedge x \neq y \wedge Tz \wedge Fxyz) \rightarrow \exists w(Pw \wedge w \neq x \wedge w \neq y \wedge Wwz))$.
Key: Px: x is a person; Tx: x is a thing; Fxyz: x and y fight for z; Wxy: x wins y.

Exercise 11

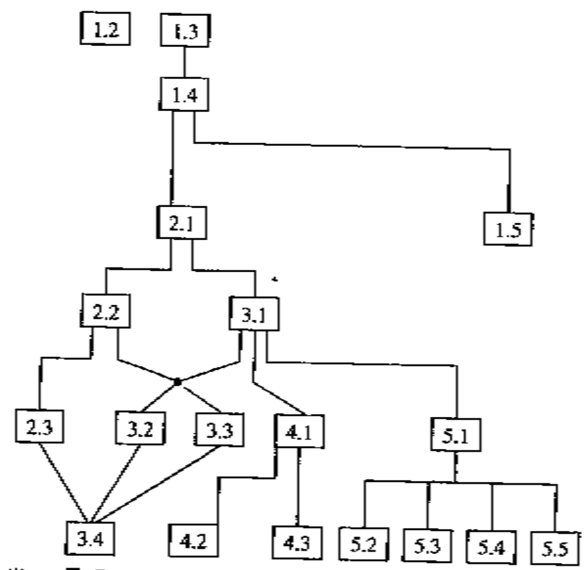
(a) No man is more clever than himself.
(b) For every man there exists another who is more clever.
(c) There is some man who is more clever than everybody except himself.
(d) There is somebody who is more clever than anybody except himself, and that is the prime minister.
(e) There are at least two queens.
(f) There are at most two queens.
(g) There are no queens except Beatrix.
(h) If two people make an exchange, then one of the two will be badly off.
(i) Any person has two parents.
(j) Mary only likes men.
(k) Charles loves no one but Elsie and Betty.
(l) Charles loves none but those loved by Betty.
(m) Nobody understands somebody who loves nobody except Mary.
(n) I help only those who help themselves.
(o) Everybody loves exactly one person.
(p) Everybody loves exactly one other person.
(q) Everybody loves a different person.
(r) All people love only themselves.
(s) People who love everybody but themselves are altruists.
(t) Altruists love each other.
(u) People who love each other are happy.

Exercise 12

(a) In many books, the dependencies between the different chapters or sections is given in the introduction by a figure. An example is figure a taken from Chang and Keisler's *Model Theory* (North-Holland, 1973). One can read figure a as a model having as its domain the set of sections {1.1, 1.3, . . . , 5.4, 5.5} in which the binary predicate letter R has been interpreted as dependency, according to the key: Rxy: y depends on x. Section 4.1, for example, depends on §3.1, but also on §§2.1, 1.4, and 1.3. For example, $\langle 2.1, 3.1\rangle \in I(R)$, and $\langle 1.4, 5.3\rangle \in I(R)$, but $\langle 2.2, 4.1\rangle \notin I(R)$.

Determine the truth values of the sentences below in the model on the basis of their meaning. Do not give all details. (With method A that is not possible anyway, since the entities in the model have not been named.)

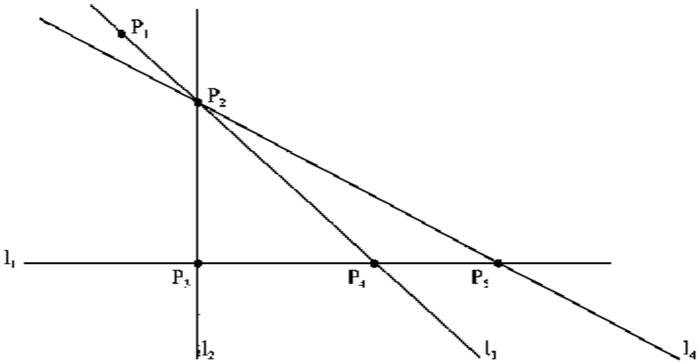
a.



(i) $\exists x Rxx$
(ii) $\exists x \exists y(x \neq y \wedge Rxy \wedge Ryx)$
(iii) $\exists x(\neg \exists y Ryx \wedge \neg \exists y Rxy)$
(iv) $\exists x \exists y(x \neq y \wedge \forall z(\neg \exists w Rzw \leftrightarrow (z = x \vee z = y)))$
(v) $\exists x \exists y \exists z(y \neq z \wedge \forall w(Rxw \leftrightarrow (w = y \vee w = z)))$
(vi) $\exists x \exists y(x \neq y \wedge \exists z Rxz \wedge \exists z Ryz \wedge \forall z(Rxz \leftrightarrow Ryz))$
(vii) $\exists x_1 \exists x_2 \exists x_3 \exists x_4 \exists x_5 \exists x_6 \exists x_7(Rx_1x_2 \wedge Rx_2x_3 \wedge Rx_3x_4 \wedge Rx_4x_5 \wedge Rx_5x_6 \wedge Rx_6x_7)$
(viii) $\forall x_1 \forall x_2 \forall x_3((x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3 \wedge \neg Rx_1x_2 \wedge \neg Rx_2x_1 \wedge \neg Rx_1x_3 \wedge \neg Rx_3x_1 \wedge \neg Rx_2x_3 \wedge \neg Rx_3x_2) \rightarrow \neg \exists y(Rx_1y \wedge Rx_2y \wedge Rx_3y))$
(ix) $\forall x \forall y((x \neq y \wedge \neg Rxy \wedge \neg Ryx) \rightarrow \neg \exists z \exists w(z \neq w \wedge \neg Rzw \wedge \neg Rwz \wedge Rxz \wedge Ryz \wedge Rxw \wedge Ryw))$

(b) Consider the model given in figure b. Its domain consists of the points and the lines in the figure. Hence $D = \{P_1, P_2, P_3, P_4, P_5, l_1, l_2, l_3, l_4\}$. The language contains the unary predicate letter P with the points as its interpretation; the unary predicate letter L with the lines as its interpretation; the binary predicate letter O with, as its interpretation, *lie on* (key: Oxy: the point x lies on the line y); and the ternary predicate letter B with, as its interpretation, *lie between* (key: Bxyz: y lies between x and z, i.e., $I(B) = \{\langle P_1, P_2, P_4\rangle, \langle P_4, P_2, P_1\rangle, \langle P_3, P_4, P_5\rangle, \langle P_5, P_4, P_3\rangle\}$).

b.



As in (a), determine the truth value in the model of the sentences below on the basis of their meaning.

(i) $\forall x(Lx \leftrightarrow \exists yOyx)$

(ii) $\forall x\forall y((Lx \wedge Ly) \rightarrow \exists z(Pz \wedge Ozx \wedge Ozy))$

(iii) $\forall x\forall y((Px \wedge Py) \rightarrow \exists z(Lz \wedge Oxz \wedge Oyz))$

(iv) $\exists x\exists y\forall z(Pz \rightarrow (Ozx \vee Ozy))$

(v) $\exists x\exists y_1\exists y_2\exists y_3\,(y_1 \neq y_2 \wedge y_1 \neq y_3 \wedge y_2 \neq y_3 \wedge \forall z((Pz \wedge Ozx) \leftrightarrow (z = y_1 \vee z = y_2 \vee z = y_3)))$

(vi) $\exists x_1\exists y_1\exists x_2\exists y_2(x_1 \neq x_2 \wedge y_1 \neq y_2 \wedge Ox_1y_1 \wedge Ox_1y_2 \wedge Ox_2y_1 \wedge Ox_2y_2)$

(vii) $\forall x\forall y\forall z(Bxyz \rightarrow Bzyx)$

(viii) $\forall x(Lx \rightarrow \exists y\exists z\exists w(Oyx \wedge Ozx \wedge Owx \wedge Byzw))$

(ix) $\forall x\forall y\forall z((x \neq y \wedge x \neq z \wedge y \neq z \wedge \exists w(Oxw \wedge Oyw \wedge Ozw)) \rightarrow (Bxyz \vee Byzx \vee Bzxy))$

(x) $\forall x(\exists y_1\exists y_2(y_1 \neq y_2 \wedge Oxy_1 \wedge Oxy_2) \rightarrow \exists z_1\exists z_2Bz_1xz_2)$

Exercise 13

There is actually a great deal of flexibility in the semantic schema presented here. Although the main emphasis has been on the case where a formula ϕ is interpreted in a given model ('verification'), there are various other modes of employment. For instance, given only some formula ϕ, one may ask for all models where it holds. Or conversely, given some model **M**, one may try to describe exactly those formulas that are true in it. And given some formulas and some nonlinguistic situation, one may even try to set up an interpretation function that makes the formulas true in that situation: this happens when we learn a foreign language. For instance, given a domain of three objects, what different interpretation functions will verify the following formula?

$$\forall x\forall y(Rxy \vee Ryx \vee x = y) \wedge \forall x\forall y(Rxy \rightarrow \neg Ryx)$$

Exercise 14 ◇

Formulas can have different numbers of models of different sizes. Show that

(i) $\exists x\forall y(Rxy \leftrightarrow \neg Ryy)$ has no models.

(ii) $\forall x \forall y(Rxy \vee Ryx \vee x = y) \wedge \forall x \forall y(Rxy \leftrightarrow \neg(Px \leftrightarrow Py))$ has only finite models of size at most two.

(iii) $\forall x \exists y Rxy \wedge \forall x \neg Rxx \wedge \exists x \forall y \neg Ryx \wedge \forall x \forall y \forall z((Rxz \wedge Ryz) \rightarrow x = y)$ has only models with infinite domains.

Exercise 15 ◇

Describe all models with finite domains of 1, 2, 3, . . . objects for the conjunction of the following formulas:

$\forall x \neg Rxx$

$\forall x \exists y Rxy$

$\forall x \forall y \forall z((Rxy \wedge Rxz) \rightarrow y = z)$

$\forall x \forall y \forall z((Rxz \wedge Ryz) \rightarrow x = y)$

Exercise 16 ◇

In natural language (and also in science), discourse often has changing domains. Therefore it is interesting to study what happens to the truth of formulas in a model when that model undergoes some transformation. For instance, in semantics, a formula is sometimes called *persistent* when its truth is not affected by enlarging the models with new objects. Which of the following formulas are generally persistent?

(i) $\exists x Px$

(ii) $\forall x Px$

(iii) $\exists x \forall y Rxy$

(iv) $\neg \forall x \forall y Rxy$

Exercise 18

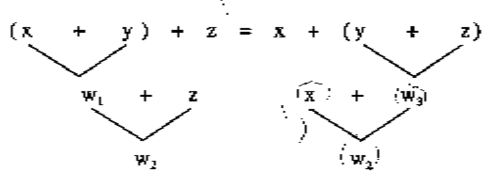
There are certain natural *operations* on binary relations that transform them into other relations. One example is *negation*, which turns a relation H into its complement, $-H$; another is *converse*, which turns a relation H into $\breve{H} = \{\langle x, y\rangle \mid \langle y, x\rangle \in H\}$. Such operations may or may not preserve the special properties of the relations defined above. Which of the following are preserved under negation or converse?

(i) reflexivity
(ii) symmetry
(iii) transitivity

3.9 Function symbols

A function is a special kind of relation. A function r from D into D can always be represented as a relation R defined as follows: $\langle d, e\rangle \in I(R)$ iff $r(d) = e$. And then $\forall x\exists!yRxy$ is true in the model in question. Conversely, if $\forall x\exists!yRxy$ is true in some model for a binary relation R, then we can define a function r which assigns the unique e such that $\langle d, e\rangle \in I(R)$ to any domain element d. So unary functions can be represented as binary relations, *n*-ary functions as $n+1$-ary relations. For example, the sum function $+$ can be represented by means of a ternary predicate letter P. Given a model with the natural numbers as its domain, we then define I(P) such that $\langle n_1, n_2, n_3\rangle \in I(P)$ iff $n_1 + n_2 = n_3$. Then, for example, $\langle 2, 2, 4\rangle \in I(P)$ and $\langle 2, 2, 5\rangle \notin I(P)$.

The commutativity of addition then amounts to the truth of $\forall x\forall y\forall z(Pxyz \rightarrow Pyxz)$ in the model. Associativity is more difficult to express. But it can be done; it is done by the following sentence: $\forall x\forall y\forall z\forall w_1\forall w_2\forall w_3((Pxyw_1 \wedge Pw_1zw_2 \wedge Pyzw_3) \rightarrow Pxw_3w_2)$. This is represented graphically in figure (106):

(106)


It is clear that expressing the properties of functions by means of predicate letters leads to formulas which are not very readable. It is for this reason that special symbols which are always interpreted as functions are often included in predicate languages, the *function symbols*.

Function symbols, like predicate letters, come in all kinds of arities: they may be unary, binary, ternary, and so forth. But whereas an *n*-ary predicate letter followed by *n* terms forms an atomic formula, an *n*-ary function symbol followed by *n* terms forms another term, an expression which refers to some entity in the domain of any model in which it is interpreted, just as constants

and variables do. Such expressions can therefore play the same roles as constants and variables, appearing in just the same positions in formulas as constants and variables do.

If the addition function on natural numbers is represented by means of the binary function symbol p, then the commutativity and associativity of addition are conveniently expressed by:

(107) $\forall x\forall y(p(x, y) = p(y, x))$

(108) $\forall x\forall y\forall z(p(p(x, y), z) = p(x, p(y, z)))$

So now we have not only simple terms like constants and variables but also composite terms which can be constructed by prefixing function symbols to the right number of other terms. For example, the expressions p(x, y), p(y, x), p(p(x, y), z), p(x, p(y, z)), and p(y, z) appearing in (107) and (108) are all composite terms. Composite terms are built up from simpler parts in much the same way as composite formulas, so they too can be given an inductive definition:

Definition 11

(i) If t is a variable or constant in L, then t is a term in L.
(ii) If f is an *n*-ary function symbol in L and $t_1, \ldots, t_n$ are terms in L, then $f(t_1, \ldots, t_n)$ is a term in L too.

The definition of the formulas of L does not have to be adapted. Their semantics becomes slightly more complicated, since we now have to begin by interpreting terms. Naturally enough, we interpret an *n*-ary function symbol f as some *n*-ary function I(f) which maps D^n, the set of all *n*-tuples of elements of the domain D of some model we are working with, into D. Variables and constants are interpreted just as before, and the interpretations of composite terms can be calculated by means of the clause:

$$[\![f(t_1, \ldots, t_n)]\!]_{M,g} = (I(f))(\langle [\![t_1]\!]_{M,g}, \ldots, [\![t_n]\!]_{M,g}\rangle).$$

So now we can see why the idea behind definition 8 is useful: it makes generalizing so much easier. In approach A, by the way, we only have to consider terms without variables, in which case $[\![t]\!]_M$ can be defined instead of $[\![t]\!]_{M,g}$.

Our account of predicate logic so far has been biased toward *predicates* of and *relations* among individual objects as the logically simple expressions. In this we followed natural language, which has few (if any) basic, i.e., lexical functional expressions. Nevertheless, it should be stressed that in many applications of predicate logic to *mathematics*, functions are the basic notion rather than predicates. (This is true, for instance, in many fields of algebra.) Moreover, at a higher level, there is much functional behavior in natural language too, as we shall see in a later chapter on type theory (see vol. 2).