(g) ∀xAx/∃x(Bx ∧ ¬Bx).
(h) ∀x∃yRxy/∃xRxx.
(i) ∀xRxx/∀x∀yRxy.
(j) ∃x∀yRxy, ∀xRxx/∀x∀y(Rxy ∨ Ryx).
(k) ∀x∃yRxy, ∀x(Rxx ↔ Ax)/∃xAx.
(l) ∀x∃yRxy, ∀x∀y(Rxy ∨ Ryx)/∀x∀y∀z((Rxy ∧ Ryz) → Rxz).
(m) ∀x∃yRxy, ∀x∀y∀z((Rxy ∧ Ryz) → Rxz)/∃xRxx.
(n) ∀x∀y(Rxy → Ryx), ∀x∀y∀z((Rxy ∧ Ryz) → Rxz)/∃xRxx.
(o) ∃x∃y∀z(x = z ∨ y = z)/∀x∀y(x = y).
(p) ∀x∃y(x ≠ y)/∃x∃y∃z(x ≠ y ∧ x ≠ z ∧ y ≠ z).
(q) ∀x∃y(Rxy ∧ x ≠ y), ∀x∀y∀z((Rxy ∧ Ryz) → Rxz)/∀x∀y(x = y ∨ Rxy ∨ Ryx).
(r) ∀x(Ax ↔ ∀yRxy), ∃x∀y(Ay ↔ x = y)/∀x∀y((Rxx ∧ Ryy) → x = y).

4.2.2 *The Principle of Extensionality*

We shall now say some more about the principle of extensionality for predicate logic and the closely related substitutivity properties, which will to some extent be proved. The following theorem, which shows a link between arguments from premises to conclusions and material implications from antecedents to consequents, will serve as an introduction:

Theorem 1

(a) $\phi \vDash \psi$ iff $\vDash \phi \rightarrow \psi$
(b) $\phi_1, \ldots, \phi_n \vDash \psi$ iff $\phi_1, \ldots, \phi_{n-1} \vDash \phi_n \rightarrow \psi$

Proof: A proof of (b) will do, since (a) is a special case of (b).

(b) ⇒: Suppose $\phi_1, \ldots, \phi_n \vDash \psi$. Suppose furthermore that for some suitable V (we shall leave out any references to the model which V originates from, if they are irrelevant) $V(\phi_1) = \ldots = V(\phi_{n-1}) = 1$. We have to show that $V(\phi_n \rightarrow \psi) = 1$ too. Suppose this is not the case. Then from the truth table for →, $V(\phi_n) = 1$ and $V(\psi) = 0$. But that is impossible, since then all of $V(\phi_1), \ldots, V(\phi_n)$ would be 1, in which case it follows from $\phi_1, \ldots, \phi_n \vDash \psi$ that $V(\psi) = 1$ and not 0.

(b) ⇐: Suppose $\phi_1, \ldots, \phi_{n-1} \vDash \phi_n \rightarrow \psi$. Suppose furthermore that for some suitable V, $V(\phi_1) = \ldots = V(\phi_n) = 1$. We have to show that then necessarily $V(\psi) = 1$. Now if $V(\phi_1) = \ldots = V(\phi_n) = 1$, then obviously $V(\phi_1) = \ldots = V(\phi_{n-1}) = 1$; according to the assumption, we then have $V(\phi_n \rightarrow \psi) = 1$, and with $V(\phi_n) = 1$ it follows that $V(\psi) = 1$. □

One direct consequence of this theorem is that in order to determine what argument schemata are valid, it is sufficient to know what formulas are universally valid. This is spelled out in theorem 2:

Theorem 2

$\phi_1, \ldots, \phi_n \vDash \psi$ iff $\vDash \phi_1 \rightarrow (\phi_2 \rightarrow (\ldots \rightarrow (\phi_n \rightarrow \psi) \ldots))$ iff $\vDash (\phi_1 \wedge \ldots \wedge \phi_n) \rightarrow \psi$.

Proof: a repeated application of theorem 1. □

There is a theorem on material equivalence which parallels theorem 1 and which we have already encountered in propositional logic.

Theorem 3

The following assertions can be deduced from each other; they are equivalent:
(i) $\phi \vDash \psi$ and $\psi \vDash \phi$
(ii) ϕ is equivalent to ψ
(iii) $\vDash \phi \leftrightarrow \psi$

Proof: It suffices to prove: (i) ⇒ (ii) ⇒ (iii) ⇒ (i).

(i) ⇒ (ii): Assume (i). Suppose, first, that $V(\phi) = 1$. Then $V(\psi) = 1$ because $\phi \vDash \psi$. Now suppose that $V(\phi) = 0$. Then it is impossible that $V(\psi) = 1$, since in that case it would follow from $\psi \vDash \phi$ that $V(\phi) = 1$, so $V(\psi) = 0$ too. Apparently $V(\phi) = V(\psi)$ under all circumstances, so that ϕ and ψ are equivalent by definition.

(ii) ⇒ (iii): Assume (ii). We now have to prove that $V(\phi \leftrightarrow \psi) = 1$ for any suitable V. But that is immediately evident, since under all circumstances $V(\phi) = V(\psi)$.

(iii) ⇒ (i): Assume (iii). Suppose now that for some V which is suitable for $\phi \vDash \psi$, $V(\phi) = 1$. Since $\phi \leftrightarrow \psi$ is universally valid, $V(\phi) = V(\psi)$ holds for all V. It follows that $V(\psi) = 1$, and we have thus proved that $\phi \vDash \psi$; $\psi \vDash \phi$ can be proved in exactly the same manner. □

This theorem can be strengthened in the same way that theorem 1 (a) is strengthened to theorem 1 (b):

Theorem 4

$(\phi_1, \ldots, \phi_n, \psi \vDash \chi$ and $\phi_1, \ldots, \phi_n, \chi \vDash \psi)$ iff $\phi_1, \ldots, \phi_n \vDash \psi \leftrightarrow \chi$.

The reader will be spared a proof.

We are now in a position to give a simple version of the promised theorem that equivalent formulas can be substituted for each other without loss of extensional meaning in predicate logic, just as in propositional logic. We shall formulate this theorem for sentences first, that is, for formulas without any free variables.

Theorem 5

If ϕ and ψ are equivalent, ϕ is a subformula of χ, and $[\psi/\phi]\chi$ is the formula obtained by replacing this subformula ϕ in χ by ψ, then χ and $[\psi/\phi]\chi$ are equivalent.

Sketch of a proof: A rigorous proof can be given by induction on (the construction of) χ. It is, however, clear (Frege's principle of compositionality!) that the truth value of ϕ has precisely the same effect on the truth value of χ as the truth value of ψ has on the truth value of $[\psi/\phi]\chi$. So if ϕ and ψ have the same truth values, then χ and $[\psi/\phi]\chi$ must too. □

The same reasoning also proves the following, stronger theorem (in which $\phi, \psi, \chi, [\psi/\phi]\chi$ are the same as above):

Theorem 6 *(Principle of extensionality for sentences in predicate logic)*

$\phi \leftrightarrow \psi \models \chi \leftrightarrow [\psi/\phi]\chi$.

And one direct consequence of theorem 6 is:

Theorem 7

If $\phi_1, \ldots, \phi_n \models \phi \leftrightarrow \psi$, then $\phi_1, \ldots, \phi_n \models \chi \leftrightarrow [\psi/\phi]\chi$.

Proof: Assume that $\phi_1, \ldots, \phi_n \models \phi \leftrightarrow \psi$. And for any suitable V, let $V(\phi_1) = \ldots = V(\phi_n) = 1$. Then of course $V(\phi \leftrightarrow \psi) = 1$. According to theorem 6 we then have $V(\chi \leftrightarrow [\psi/\phi]\chi) = 1$, whence $\phi_1, \ldots, \phi_n \models \chi \leftrightarrow [\psi/\phi]\chi$ is proved. □

Theorem 7 can be paraphrased as follows: if two sentences are equivalent (have the same extensional meaning) under given assumptions, then under the same assumptions, they may be substituted for each other without loss of extensional meaning. There is also a principle of extensionality for formulas in general; but first we will have to generalize theorem 3 so that we can use the equivalence of formulas more easily.

Theorem 8

If the free variables in ϕ and in ψ are all among $x_1, \ldots, x_n$, then ϕ and ψ are equivalent iff $\models \forall x_1 \ldots \forall x_n(\phi \leftrightarrow \psi)$.

Proof: The proof will only be given for $n = 1$, since the general case is not essentially different. We will write x for x_1.

$\Rightarrow$: Suppose ϕ and ψ are equivalent. Then by definition, for every suitable **M** and g, $V_{M,g}(\phi) = V_{M,g}(\psi)$. That is, for every suitable **M**

and g, $V_{M,g}(\phi \leftrightarrow \psi) = 1$. But then, for every suitable **M**, g, and $d \in D_M$, $V_{M,g[x/d]}(\phi \leftrightarrow \psi) = 1$. According to Tarski's truth definition, this means that for every suitable M and g, $V_{M,g}(\forall x(\phi \leftrightarrow \psi)) = 1$. And this is the conclusion we needed.

$\Leftarrow$: The above proof of $\Rightarrow$ also works in reverse. □

We can now prove a principle of extensionality for formulas in predicate logic, just as we proved theorems 6 and 7. We give the theorems and omit their proofs. The conditions on ϕ, ψ, χ, and $[\psi/\phi]\chi$ are the same as above except that ϕ and ψ may now be formulas, with the proviso that their free variables are all among $x_1, \ldots, x_n$ (if ϕ and ψ are sentences, then $n = 0$).

Theorem 9 *(Principle of extensionality for predicate logic)*

$\forall x_1 \ldots \forall x_n(\phi \leftrightarrow \psi) \models \chi \leftrightarrow [\psi/\phi]\chi$

Theorem 10

If $\phi_1, \ldots, \phi_m \models \forall x_1 \ldots \forall x_n(\phi \leftrightarrow \psi)$, then $\phi_1, \ldots, \phi_m \models \chi \leftrightarrow [\psi/\phi]\chi$

Theorem 10 again expresses the fact that formulas with the same extensional meaning can be substituted for each other without loss of extensional meaning. Actually this theorem sanctions, for example, leaving off the brackets in conjunctions and disjunctions with more than two members (see §2.5). Theorems 9 and 10 can be generalized so that ϕ need not have precisely the variables $x_1, \ldots, x_n$ in χ. A more general formulation is, however, somewhat tricky, and for that reason will not be given.

We conclude our discussion of the principle of extensionality for predicate logic with a few examples. The formulas $\forall x(Ax \wedge Bx)$ and $\forall xAx \wedge \forall xBx$ are equivalent. From this it follows from theorem 3 that $\forall x(Ax \wedge Bx) \models \forall xAx \wedge \forall xBx$, that $\forall xAx \wedge \forall xBx \models \forall x(Ax \wedge Bx)$, and that $\models \forall x(Ax \wedge Bx) \leftrightarrow (\forall xAx \wedge \forall xBx)$. This last can be used for theorem 10, with n = 0. If we choose $\forall x(Ax \wedge Bx) \rightarrow \exists x \neg Cx$ as our χ, then it follows that $\forall x(Ax \wedge Bx) \rightarrow \exists x \neg Cx$ and $(\forall xAx \wedge \forall xBx) \rightarrow \exists x \neg Cx$ are equivalent. And so on.

The equivalence of $Ax \wedge Bx$ and $Bx \wedge Ax$ results, using theorem 8, in $\models \forall x((Ax \wedge Bx) \leftrightarrow (Bx \wedge Ax))$. Applying theorem 10 to this, we obtain the equivalence of $\forall x((Ax \wedge Bx) \rightarrow \exists yRxy)$ and $\forall x((Bx \wedge Ax) \rightarrow \exists yRxy)$. Equivalences other than the commutativity of $\wedge$ can also be applied, the associative laws for $\wedge$ and $\vee$, for example, which result in the fact that in predicate logic as in propositional logic, brackets can be left out both in strings of conjunctions and in strings of disjunctions. Here is an application of theorem 10 with m > 0: it is not difficult to establish that $\neg(\exists xAx \wedge \exists xBx) \models \forall x(Ax \vee Bx) \leftrightarrow (\forall xAx \vee \forall xBx)$. It follows that $\neg(\exists xAx \wedge \exists xBx) \models (\forall xCx \rightarrow \forall x(Ax \vee Bx)) \leftrightarrow (\forall xCx \rightarrow (\forall xAx \vee \forall xBx))$, to take just one arbitrary example.

Given the above, we are also in a position to say more about problems with extralogical meanings, which we have noticed in connection with pairs of sentences like (8) (= (2)):

(8) Casper is bigger than Peter
Peter is smaller than Casper

Having translated *x is bigger than y* into predicate logic as Bxy, and *x is smaller than y* as Sxy, we now take $\forall x \forall y(Bxy \leftrightarrow Syx)$ as a permanent assumption, since we are only interested in models **M** in which $V_M(\forall x \forall y(Bxy \leftrightarrow Syx)) = 1$. Under this assumption, Bxy and Syx are equivalent. Furthermore, according to theorem 10, Bzw and Swz are equivalent for arbitrary variables z and w, since $\forall x \forall y(Bxy \leftrightarrow Syx) \models \forall z \forall w(Bzw \leftrightarrow Swz)$. In fact, it is not too difficult to see that Bt_1t_2 and St_2t_1 are also equivalent for arbitrary terms t_1 and t_2, as in Ba_1a_2 and Sa_2a_1, for example, so that if *Casper* is translated as a_1 and *Peter* as a_2, both of the sentences in (8) have the same extensional meaning. An assumption like the one we are discussing is called a *meaning postulate*. The problem with (9) (=(3)):

(9) Pierre is a bachelor.
Pierre is an unmarried man.

can be resolved in much the same manner by taking $\forall x((Mx \land \neg Wx) \leftrightarrow Bx)$ as our meaning postulate; the key to the translation is Bx: x is a bachelor; Wx: x is married; Mx: x is a man. What meaning postulates do is provide information about what words mean. They are comparable with dictionary definitions in which *bachelor*, for example, is defined as *unmarried man*. In mathematics, some axioms play the role of meaning postulates. For instance, the following axioms relate the meanings of some key notions in geometry. If we interpret Px as x is a point; Lx as x is a line; and Oxy as x lies on y, for example, the following geometrical axioms can be drawn up: $\forall x \forall y((Px \land Py \land x \neq y) \rightarrow \exists! z(Lz \land Oxz \land Oyz))$, that is, given two different points, exactly one line can be drawn which passes through both, and $\forall x \forall y((Lx \land Ly \land x \neq y) \rightarrow \forall z \forall w((Pz \land Pw \land Ozx \land Ozy \land Owx \land Owy) \rightarrow z = w))$, that is, two different lines have at most one point in common.

In addition to the principles discussed above, there are also principles of extensionality dealing with constants and variables, not in connection with truth values, of course, but in terms of elements in a domain. Constants, and variables too, by assignments, are interpreted as elements in a domain. Here are two examples of such theorems, without proofs:

Theorem 11

If s and t are terms lacking variables, then for the formula $[t/s]\phi$ obtained by substituting t for s in ϕ, we have: $s = t \models \phi \leftrightarrow [t/s]\phi$.

Theorem 12

If s_1, s_2, and t are terms whose variables are all among $x_1, \ldots, x_n$, then for the term $[s_2/s_1]t$ obtained by substituting s_2 for s_1 in t, we have: $\models \forall x_1 \ldots \forall x_n(s_1 = s_2 \rightarrow [s_2/s_1]t = t)$.

Here are some applications of these theorems, in a language with p as a binary function symbol for the addition function: $a_4 = p(a_2, a_2) \models p(a_4, a_4) = p(p(a_2, a_2), p(a_2, a_2))$, and $\models \forall x \forall y \forall z(p(x, y) = p(y, x) \rightarrow p(p(x, y), z) = p(p(y, x), z))$.

We conclude this section by returning briefly to what we said in § 1.1: that substituting sentences for the variables of a valid argument schema is supposed to result in another valid argument schema. Predicate logic does indeed comply with this: substituting formulas for the predicate letters in valid argument schemata results in other, valid argument schemata. But there are complications having to do with bound and free variables which mean that restrictions have to be placed on the substitutions, so that giving a general formulation is difficult. We will just give an example: the substitution of predicate-logical formulas in purely propositional argument schemata:

Theorem 13

Assume that $\phi_1, \ldots, \phi_n \models \psi$ in propositional logic and that $\phi_1, \ldots, \phi_n$ and ψ contain no propositional letters except $p_1, \ldots, p_m$. And let $\chi_1, \ldots, \chi_m$ be sentences in some predicate-logical language L, while $\phi_1', \ldots, \phi_n'$ and ψ' are obtained from $\phi_1, \ldots, \phi_n$ and ψ by (simultaneously) substituting $\chi_1, \ldots, \chi_m$ for $p_1, \ldots, p_m$. Then $\phi_1', \ldots, \phi_n' \models \psi'$ in predicate logic.

Proof: Suppose that $\phi_1, \ldots, \phi_n \models \psi$, but $\phi_1', \ldots, \phi_n' \not\models \psi'$. Then there is a counterexample **M** which is responsible for the latter: $V_M(\phi_1') = \ldots = V_M(\phi_n') = 1$ and $V_M(\psi') = 0$. Then a propositional counterexample to the former argument schema can be obtained by taking: $V(p_i) = V_M(\chi_i)$ for every i between 1 and m. Then it is clear that $V(\phi_1) = \ldots = V(\phi_n) = 1$ but that $V(\psi) = 0$, since $\phi_1, \ldots, \phi_n$ and ψ are composed of $p_1, \ldots, p_m$ in exactly the same way as $\phi_1', \ldots, \phi_n'$ and ψ' are composed of $\chi_1, \ldots, \chi_m$. We now have a counterexample to our first assumption $\phi_1, \ldots, \phi_n \models \psi$, so it cannot be the case that $\phi'_1, \ldots, \phi'_n \not\models \psi'$. □

One simple consequence of theorem 13 is that substitution instances of propositional tautologies are universally valid formulas. Here are a few applications:

(a) $p \land q \models q \land p$, so, for example,
$(r \lor s) \land (p \rightarrow q) \models (p \rightarrow q) \land (r \lor s)$ and
$\forall x \exists y Axy \land \forall x \exists y Bxy \models \forall x \exists y Bxy \land \forall x \exists y Axy$

(b) $\models ((p \rightarrow q) \rightarrow p) \rightarrow p$, so, for example,
$\models (((p \rightarrow q) \rightarrow (q \rightarrow r)) \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q)$ and
$\models ((\forall xAx \rightarrow \exists yBy) \rightarrow \forall xAx) \rightarrow \forall xAx$.

We conclude this section with an example of an argument schema drawn from predicate logic, for which formulating a general theorem like the above takes too much doing:

(c) $\forall x(Ax \vee Bx), \exists x\neg Ax \models \exists xBx$, so, for example,
$\forall x((Ax \wedge Bx) \vee (Ax \wedge Cx)), \exists x\neg(Ax \wedge Bx) \models \exists x(Ax \wedge Cx)$ and
$\forall x(\exists yAxy \vee \exists zBxz), \exists x\neg\exists yAxy \models \exists x\exists zBxz$.