each other.

5.2 Definite Descriptions

In standard predicate logic there is just one kind of expression which can be used to refer to some entity or individual in particular, and that is the individual constant. The whole idea behind individual variables is that they do not refer to particular individuals but can be used to refer to various different things. In just about all of the examples of translations from natural language into predicate logic which we have seen so far, individual constants have served as translations of proper names. Proper names are expressions which refer to particular individual things, but fortunately they are not the only expressions which can be used for this purpose. If they were, it would be impossible to talk about people without knowing their names. Another way of referring to a particular individual or thing is by means of a description, as in (1)–(4).

(1) The queen of the Netherlands

(2) The first man on the moon

(3) Elvis Presley's mother

(4) Ronald Reagan's ranch

Expressions like these are called *definite descriptions*. With the exception of (4), each of the examples comprises a predicate expression, which may be composite, and a definite article. And the possessive *Ronald Reagan's* in (4) can be seen as a composite definite article. The predicates in the examples have been chosen so that we can be reasonably sure that there is just one individual who satisfies them, and these are then the unique individuals to whom the definite descriptions refer.

So far we have just used individual constants as the formal translations of definite descriptions. But the translations become more true if we introduce a special notation for them which does justice to the fact that they are composite expressions. For this purpose we now introduce the *iota operator* ℩ (an upside-down Greek iota) which, like the existential and universal quantifiers, always comes with a variable and is always followed by a propositional function which is its scope. Thus it appears in expressions like ℩xFx, ℩xGxy, and ℩x(Fx ∧ Gxa). We call such expressions *descriptions*. Descriptions are complex terms, since while a quantifier followed by a propositional function is a sentence or another propositional function, the iota operator followed by a propositional function is always a term, an expression which can appear among the arguments of an *n*-ary predicate just like an individual constant or variable. So we obtain formulas like:

(5)	R(℩xQx)	The queen of the Netherlands is riding a bicycle.
(6)	b = ℩xQx	Beatrix is the queen of the Netherlands.
(7)	℩xQx = ℩xHx	The queen is the head of state.
(8)	∀x(Dx → L(x, ℩yQy))	Every Dutchman loves the queen.
(9)	w = ℩xS(x, ℩yQy)	Willem-Alexander is the queen's son.

Although it is strictly speaking unnecessary, we shall on occasion add extra brackets and separate the arguments of relations by means of commas, thus making the formulas more readable. Note also that in these examples the expression *queen of the Netherlands*, among others, has been rendered as a unary predicate. We could, of course, preserve more structure by translating (5), for example, as R(℩xQ(x, n)).

In order to incorporate the descriptions formed with the iota operator into the language of predicate logic, we must expand the definition of the formulas of predicate logic (definition 1 in §3.3) to a *simultaneous inductive definition of both terms and formulas*. We have to define both together because formulas can now be among the parts from which a term is built up, and vice versa. Here are the clauses which must be added in order to achieve this:

(a) If α is an individual constant or variable in L, then α is a term in L.
(b) If ϕ is a formula in L and x is a variable, then $\iota x\phi$ is a term in L.

The clause giving the atomic formulas is then:

(i) if A is an n-ary predicate letter ($n \geq 1$) and $t_1 \ldots t_n$ are terms in L, then $At_1 \ldots t_n$ is a formula in L.

Clauses (ii)–(iv) for connectives and quantifiers do not need modification. Only the final clause still needs to be adapted:

(v) Only that which can be generated by clauses (i)–(iv) in a finite number of steps is a formula or term in L.

The syntactic innovation obtained by introducing the iota operator into the language of predicate logic is not sufficient. We also have to adjust the semantics to fit, saying how the new descriptions are to be interpreted. Here we use the approach to interpretation given as B in §3.6.3, which makes use of assignments. We now join definition 8, which interprets terms, with definition 9, the truth definition, thus obtaining a new definition that simultaneously interprets both terms and formulas of the language for predicate logic with descriptions. In order to interpret descriptions we add the following new clause:

(10) $[\![\iota x\phi]\!]_{M,g}$ is the unique individual $d \in D$ such that $V_{M,g[x/d]}(\phi) = 1$.

We must link the definitions interpreting terms with those interpreting formulas in this way because the interpretation of any term is now dependent on the interpretations of the formulas appearing in it (and vice versa). The problem with (10), however, is that $[\![\iota x\phi]\!]_{M,g}$ is not defined if there isn't exactly one individual satisfying ϕ. If there is no such individual, or if there are too many, then (10) does not say how $\iota x\phi$ should be interpreted. As examples of descriptions where this goes wrong in the real world, we have (11) and (12). Example (13) is a well-known example due to Russell.

(11) Queen Beatrix's brother

(12) Queen Beatrix's daughter

(13) the king of France

The fact that these descriptions are undefined also transfers to some sentences in which they appear. Sentence (14), for example, is neither true nor false:

(14) The king of France is bald.

To put this formally, if there is no unique individual that satisfies ϕ, then not only $[\![\iota x\phi]\!]_{M,g}$ but also $V_{M,g}(F(\iota x\phi))$ is undefined, which means that the for-

mula $F(\iota x\phi)$ is neither true nor false. But this is not allowed by the fundamental principle of bivalence, which requires every formula to be either true or false. There are various ways this problem can be dealt with, and we shall only discuss here the solutions given by Frege and Russell. They have in common that they both strive to maintain the principle of bivalence. In this their approach differs from that taken in many-valued logic, where more truth values are considered than just *true* and *false*. We shall return to this approach later in this section, and at greater length in §5.5, which is devoted to many-valued logic.

Frege saw as a shortcoming of natural language the occurrence of definite descriptions which do not denote some unique thing. A properly constructed logical language, he thought, should always provide some unique descriptum. One way of doing this is to include a special *nil entity* in the domain, which is then by convention taken to be the entity denoted by descriptions which fail to satisfy the existential requirement or the requirement of unicity. The same thing is done in mathematics, where, for example, 0 is taken as the value of x/0 if it is desired that x/y always be defined. It is clear that Frege's solution is purely formal and not very intuitive. But it does solve the technical difficulties.

If d_0 is the special nil individual, then the clause-interpreting descriptions can be as follows:

(15) $[\![\iota x\phi]\!]_{M,g}$ is the unique individual $d \in D$ such that $V_{M,g[x/d]}(\phi) = 1$ if there is any such thing; otherwise it is the nil individual d_0.

Given (15), the interpretation of descriptions is defined under all circumstances. And if we make sure that d_0 does not belong to the interpretation of any normal predicates such as *bald*, then sentence (14) is false.

The solution given by Russell has in common with that proposed by Frege not only that the principle of bivalence is maintained but also that a shortcoming of natural language is seen as the root of the problem. Russell's solution is known as his *theory of descriptions* and was first presented in his article "On Denoting" (1905). The approach is in line with the *misleading form thesis*, according to which the grammatical form of sentences sometimes does not reflect their 'real' logical form and is as a result misleading (see also §1.5.1). This thesis has played a prominent role in analytic philosophy. To get past the superficial grammatical form of sentences and reveal their underlying logical form was taken to be an important task for philosophy, and Russell's theory of descriptions is a textbook example of an attempt to do this.

In analyzing definite descriptions as descriptions formed by means of the iota operator, we have assumed that definite descriptions and proper names have the same syntactic function. Sentences like (16) and (17) would seem to suggest that this is reasonable enough:

(16) Beatrix is riding a bicycle.

(17) The queen of the Netherlands is riding a bicycle.

Both the proper name *Beatrix* and the definite description *the queen of the Netherlands* seem to fit the role of the subject of the predicate *is riding a bicycle*. This is where Russell would interrupt, saying that the grammatical form of these two sentences is misleading. Definite descriptions should not be considered normal subjects any more than quantified expressions like *everyone* and *no one*. The problems with definite descriptions result from our mistaking their misleading grammatical form for their logical form.

Russell's theory of descriptions provides us with a method for translating formulas containing the iota operator into formulas containing only the familiar quantifiers of standard predicate logic. This method uses *contextual definitions*. We cannot give a general definition of the iota operator and the descriptions formed with it (which would be an *explicit definition*). But for any given formula containing a description, that is, in any particular context, we can give an equivalent formula in which the iota operator is replaced by the normal quantifiers. The elimination of the iota operator means that the principle of bivalence can be maintained. According to Russell, a sentence like (17) says that there is an individual x who has the following three properties:

(i) x is queen of the Netherlands: Qx;
(ii) there is no individual y besides x that has the property of being queen of the Netherlands: $\forall y(Qy \rightarrow y = x)$; and
(iii) x is riding a bicycle: Rx.

This means that sentence (17) can be translated as the following formula; it may seem a bit complicated but is in standard predicate logic:

(18) $\exists x(Qx \wedge \forall y(Qy \rightarrow y = x) \wedge Rx)$

Or equivalently and a little more simply:

(19) $\exists x(\forall y(Qy \leftrightarrow y = x) \wedge Rx)$

In general, the above means that every formula of the form $G(\iota xFx)$ can be reduced to a formula in standard predicate logic by means of the following definition:

Definition 1

$G(\iota xFx) =_{def} \exists x(\forall y(Fy \leftrightarrow y = x) \wedge Gx)$

As we have said, this is a contextual definition of descriptions. The iota operator cannot be given an explicit definition in predicate logic. Note also that definition 1 can be made more general, since as it stands it can only be used if the propositional function in the description is an atomic formula with a unary predicate and if the context is such that description itself appears as the argument of a unary predicate. The obvious general formulation of definition 1 will be omitted here.

Sentences like (14), with descriptions which fail to satisfy the existential

requirement or which fail to satisfy the requirement of unicity, are simply false, as was the case in Frege's analysis. If the Netherlands were not a monarchy or simply did not have a monarch, then (19), the translation of (17), would be false. This is the strength of Russell's theory, but according to some, like Strawson (1950), it is also its weakness. According to Strawson's analysis, the existence of exactly one individual having the property of being queen of the Netherlands is not *stated* when sentence (17) is uttered; it is *presupposed*. And if this *presupposition* is not satisfied, then we cannot say that a proposition is being expressed which is either true or false. We shall not attempt to say who was right, Strawson or Russell. More important for our purposes here are some of the implications of Strawson's position from a logical point of view. Russell's treatment of definite descriptions leaves standard predicate logic untouched, but Strawson's approach would seem to challenge the principle of bivalence. In §5.5 we shall see some of the attempts to give Strawson's position a logical basis by means of a system of many-valued logic.

Any theory of definite descriptions has to give some account of negative expressions containing definite descriptions like the following:

(20) The queen of the Netherlands is not riding a bicycle.

(21) The king of France is not bald.

For Strawson, the question is quite simple: these sentences presuppose the existence of a unique queen of the Netherlands and a unique king of France, just as do the positive sentences we started with, and state that the former is not riding a bicycle and the latter is not bald.

Russell's theory is a little more subtle. Superficially one might think that sentence (21) is just the negation of sentence (14), so that it must be true under any circumstances under which (14) is false. According to Russell, it is not so simple. He takes a sentence like (21) as ambiguous, with one reading in which it is true and another in which, like (14), it is false. The reading in which it is true can be paraphrased as: it is not the case that there is a unique individual who is king of France and who is bald. Formula (22) corresponds to this reading. The reading in which (21) is false can be paraphrased as: there is a unique individual who is king of France and is not bald. Corresponding to this reading we have formula (23) (Kx: x is king of France; Bx: x is bald).

(22) $\neg\exists x(\forall y(Ky \leftrightarrow y = x) \wedge Bx)$

(23) $\exists x(\forall y(Ky \leftrightarrow y = x) \wedge \neg Bx)$

Both of these standard predicate-logical formulas can be obtained from the representation of (21) by means of the iota operator: $\neg B(\iota xKx)$. The first is obtained by applying definition 1 to $B(\iota xKx)$ in the formula $\neg B(\iota xKx)$. This gives the negation operator $\neg$ wide scope over the quantifiers, as is apparent from (22). Formula (23) is obtained by applying definition 1 to the formula

¬B(ɿxKx) itself. In this case the quantifiers have wide scope. In Russell's own terminology, (23) represents the reading of (21) in which the definite description has a *primary* occurrence, and (22) represents the reading in which the definite description has a *secondary* occurrence.

In Frege's approach, a sentence like (21) is most naturally given just the reading in which it is untrue. But even if descriptions are given a Fregean interpretation, it is still possible to translate them into the normal quantifiers by means of a contextual definition. And if this is done, then an ambiguity arises which is similar to the one we saw with Russell. The advantage which Russell's theory has over Frege's theory is that it does not need any nil entity. Frege's theory, on the other hand, enables definite descriptions to be interpreted as such. We have mentioned that Russell's theory of descriptions is inspired by the idea that grammatical form is often misleading. From a syntactic point of view, definite descriptions would seem to be able to play the same role as proper names; they would seem to be independent entities. But apparently this is not true from a logical point of view. The fact that descriptions only admit of a contextual definition shows that, at least as far as their logical form is concerned, definite descriptions are not independent entities. There is no logical expression corresponding to the description *the queen of the Netherlands*. In this way, descriptions resemble (other) quantified terms like *every man, some men,* and *all men*. The logical form of expressions like this can only be given relative to the contexts, the whole sentences, in which they appear. Like the logical analysis of universally and existentially quantified sentences, Russell's theory of descriptions would seem to support the idea that there is a fundamental difference between the grammatical, that is, the surface syntactic form of sentences, and their logical form. It is an idea which has been extremely influential.

Note that all the talk here about 'logical expressions' and 'logical form' is really just about expressions in standard predicate logic and standard predicate-logical form. And our conclusion that there is an essential difference between grammatical form and logical form must be read with this restriction in mind. Descriptions and quantifiers may not be independent units from the perspective of predicate logic, but that is not to say that there are no logical systems in which they are independent units. We shall show in volume 2 that both definite descriptions and (other) quantified expressions can be translated into the formal language if we consider a richer logical language than that of standard predicate logic (namely, higher-order logic with lambda abstraction), so that they can be interpreted as independent units. And in that way descriptions and quantified expressions can also be placed in the same logical category, so that the grammatical form which Russell considered so misleading can, as far as logical form is concerned, be rehabilitated. These results have argued against the influential idea that there is a fundamental distinction between grammatical and logical form.