SECTION 1.4

Induction and Recursion[1]

Induction

There is one special type of construction that occurs frequently both in logic and in other branches of mathematics. We may want to construct a certain subset of a set U by starting with some initial elements of U, and applying certain operations to them over and over again. The set we seek will be the smallest set containing the initial elements and closed under the operations. Its members will be those elements of U which can be built up from the initial elements by applying the operations some finite number of times.

[1] On the one hand, the concepts in this section are important, and they arise in many places throughout mathematics. On the other hand, readers may want to postpone — not skip — study of this section.

In the special case of immediate interest to us, U is the set of expressions, the initial elements are the sentence symbols, and the operations are $\mathcal{E}_\neg$, $\mathcal{E}_\wedge$, etc. The set to be constructed is the set of wffs. But we will encounter other special cases later, and it will be helpful to view the situation abstractly here.

To simplify our discussion, we will consider an initial set $B \subseteq U$ and a class $\mathcal{F}$ of functions containing just two members f and g, where

$$f : U \times U \to U \quad \text{and} \quad g : U \to U.$$

Thus f is a binary operation on U and g is a unary operation. (Actually $\mathcal{F}$ need not be finite; it will be seen that our simplified discussion here is, in fact, applicable to a more general situation. $\mathcal{F}$ can be any set of relations on U, and in Chapter 2 this greater generality will be utilized. But the case discussed here is easier to visualize and is general enough to illustrate the ideas. For a less restricted version, see Exercise 3.)

If B contains points a and b, then the set C we wish to construct will contain, for example,

$$b, \quad f(b, b), \quad g(a), \quad f(g(a), f(b, b)), \quad g(f(g(a), f(b, b))).$$

Of course these might not all be distinct. The idea is that we are given certain bricks to work with, and certain types of mortar, and we want C to contain just the things we are able to build.

In defining C more formally, we have our choice of two definitions. We can define it "from the top down" as follows: Say that a subset S of U is *closed* under f and g iff whenever elements x and y belong to S, then so also do $f(x, y)$ and $g(x)$. Say that S is *inductive* iff $B \subseteq S$ and S is closed under f and g. Let $C^\star$ be the intersection of all the inductive subsets of U; thus $x \in C^\star$ iff x belongs to every inductive subset of U. It is not hard to see (and the reader should check) that $C^\star$ is itself inductive. Furthermore, $C^\star$ is the smallest such set, being included in all the other inductive sets.

The second (and equivalent) definition works "from the bottom up." We want $C_\star$ to contain the things that can be reached from B by applying f and g a finite number of times. Temporarily define a *construction sequence* to be a finite sequence $\langle x_1, \ldots, x_n \rangle$ of elements of U such that for each $i \leq n$ we have at least one of

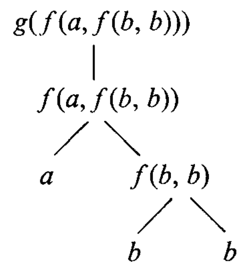
$$\begin{aligned} &x_i \in B, && \\ &x_i = f(x_j, x_k) && \text{for some} \quad j < i, k < i, \\ &x_i = g(x_j) && \text{for some} \quad j < i. \end{aligned}$$

In other words, each member of the sequence either is in B or results from *earlier* members by applying f or g. Then let $C_\star$ be the set of all points x such that some construction sequence ends with x.

Let C_n be the set of points x such that some construction sequence of length n ends with x. Then $C_1 = B$,

$$C_1 \subseteq C_2 \subseteq C_3 \subseteq \cdots,$$

and $C_\star = \bigcup_n C_n$. For example, $g(f(a, f(b, b)))$ is in C_5 and hence in $C_\star$, as can be seen by contemplating the tree shown:



We obtain a construction sequence for $g(f(a, f(b, b)))$ by squashing this tree into a linear sequence.

Examples

1. The natural numbers. Let U be the set of all real numbers, and let $B = \{0\}$. Take one operation S, where $S(x) = x + 1$. Then

$$C_\star = \{0, 1, 2, \ldots\}.$$

The set $C_\star$ of natural numbers contains exactly those numbers obtainable from 0 by applying the successor operation repeatedly.

2. The integers. Let U be the set of all real numbers; let $B = \{0\}$. This time take two operations, the successor operation S and the predecessor operation P:

$$S(x) = x + 1 \quad \text{and} \quad P(x) = x - 1.$$

Now $C_\star$ contains all the integers,

$$C_\star = \{\ldots, -2, -1, 0, 1, 2, \ldots\}.$$

Notice that there is more than one way of obtaining 2 as a member of $C_\star$. For 2 is $S(S(0))$, but it is also $S(P(S(S(0))))$.

3. The algebraic functions. Let U contain all functions whose domain and range are each sets of real numbers. Let B contain the identity function and all constant functions. Let $\mathcal{F}$ contain the operations (on functions) of addition, multiplication, division, and root extraction. Then $C_\star$ is the class of algebraic functions.

4. The well-formed formulas. Let U be the set of all expressions and let B be the set of sentence symbols. Let $\mathcal{F}$ contain the five formula-building operations on expressions: $\mathcal{E}_\neg$, $\mathcal{E}_\wedge$, $\mathcal{E}_\vee$, $\mathcal{E}_\rightarrow$, and $\mathcal{E}_\leftrightarrow$. Then $C_\star$ is the set of all wffs.

At this point we should verify that our two definitions are actually equivalent, i.e., that $C^\star = C_\star$.

To show that $C^\star \subseteq C_\star$ we need only check that $C_\star$ is inductive, i.e., that $B \subseteq C_\star$ and $C_\star$ is closed under the functions. Clearly $B = C_1 \subseteq C_\star$. If x and y are in $C_\star$, then we can concatenate their construction sequences and append $f(x, y)$ to obtain a construction sequence placing $f(x, y)$ in $C_\star$. Similarly, $C_\star$ is closed under g.

Finally, to show that $C_\star \subseteq C^\star$ we consider a point in $C_\star$ and a construction sequence $\langle x_0, \ldots, x_n \rangle$ for it. By ordinary induction on i, we can see that $x_i \in C^\star$, $i \leq n$. First $x_0 \in B \subseteq C^\star$. For the inductive step we use the fact that $C^\star$ is closed under the functions. Thus we conclude that

$$\bigcup_n C_n = C_\star = C^\star = \bigcap \{S \mid S \quad \text{is inductive}\}.$$

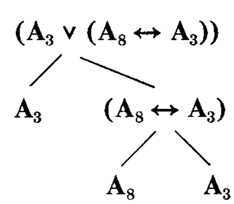
(A parenthetical remark: Suppose our present study is embedded in axiomatic set theory, where the natural numbers are usually defined from the top down. Then our definition of $C_\star$ (employing finiteness and hence natural numbers) is not really different from our definition of $C^\star$. But we are not working within axiomatic set theory; we are working within informal mathematics. And the notion of natural number seems to be a solid, well-understood intuitive concept.)

Since $C^\star = C_\star$, we call the set simply C and refer to it as the set *generated from B by* the functions in $\mathcal{F}$. We will often want to prove theorems by using the following:

Induction Principle Assume that C is the set generated from B by the functions in $\mathcal{F}$. If S is a subset of C that includes B and is closed under the functions of $\mathcal{F}$ then $S = C$.

Proof. S is inductive, so $C = C^\star \subseteq S$. We are given the other inclusion. ⊣

The special case now of interest to us is, of course, Example 4. Here C is the class of wffs generated from the set of sentence symbols by the formula-building operations. This special case has interesting features of its own. Both α and β are proper segments of $\mathcal{E}_\wedge(\alpha, \beta)$, i.e., of $(\alpha \wedge \beta)$. More generally, if we look at the family tree of a wff, we see that each constituent is a proper segment of the end product.



Suppose, for example, that we temporarily call an expression *special* if the only sentence symbols in it are among $\{\mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_5\}$ and the only connective symbols in it are among $\{\neg, \rightarrow\}$. Then no special wff requires $\mathbf{A}_9$ or $\mathcal{E}_\wedge$ for its construction. In fact, every special wff belongs to the set C_s generated from $\{\mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_5\}$ by $\mathcal{E}_\neg$ and $\mathcal{E}_\rightarrow$. (We can use the induction principle to show that every wff either belongs to C_s or is not special.)

Recursion

We return now to the more abstract case. There is a set U (such as the set of all expressions), a subset B of U (such as the set of sentence symbols), and two functions f and g, where

$$f : U \times U \to U \quad \text{and} \quad g : U \to U.$$

C is the set generated from B by f and g.

The problem we now want to consider is that of defining a function on C recursively. That is, we suppose we are given

1. Rules for computing $\overline{h}(x)$ for $x \in B$.

2a. Rules for computing $\overline{h}(f(x, y))$, making use of $\overline{h}(x)$ and $\overline{h}(y)$.

2b. Rules for computing $\overline{h}(g(x))$, making use of $\overline{h}(x)$.

(For example, this is the situation discussed in Section 1.2, where $\overline{h}$ is the extension of a truth assignment for B.) It is not hard to see that there can be at most one function $\overline{h}$ on C meeting all the given requirements.

But it is possible that no such $\overline{h}$ exists; the rules may be contradictory. For example, let

$$\begin{aligned} U &= \text{the set of real numbers}, \\ B &= \{0\}, \\ f(x, y) &= x \cdot y, \\ g(x) &= x + 1. \end{aligned}$$

Then C is the set of natural numbers. Suppose we impose the following requirements on $\overline{h}$:

1. $\overline{h}(0) = 0$.

2a. $\overline{h}(f(x, y)) = f(\overline{h}(x), \overline{h}(y))$.

2b. $\overline{h}(g(x)) = \overline{h}(x) + 2$.

Then no such function $\overline{h}$ can exist. (Try computing $\overline{h}(1)$, noting that we have both $1 = g(0)$ and $1 = f(g(0), g(0))$.)

A similar situation is encountered in algebra.[2] Suppose that you have a group G that is generated from B by the group multiplication

[2] It is hoped that examples such as this will be useful to the reader with some algebraic experience. The other readers will be glad to know that these examples are merely illustrative and not essential to our development of the subject.

and inverse operation. Then an arbitrary map of B into a group H is not necessarily extendible to a homomorphism of the entire group G into H. But if G happens to be a free group with set B of independent generators, then any such map is extendible to a homomorphism of the entire group.

Say that C is *freely* generated from B by f and g iff in addition to the requirements for being generated, the restrictions f_C and g_C of f and g to C meet the following conditions:

1. f_C and g_C are one-to-one.
2. The range of f_C, the range of g_C, and the set B are pairwise disjoint.

The main result of this section, the recursion theorem, says that if C is freely generated then a function h on B always has an extension $\overline{h}$ on C that follows the sorts of rules considered above.

Recursion Theorem Assume that the subset C of U is freely generated from B by f and g, where

$$f : U \times U \to U,$$
$$g : U \to U.$$

Further assume that V is a set and F, G, and h functions such that

$$h : B \to V,$$
$$F : V \times V \to V,$$
$$G : V \to V.$$

Then there is a unique function

$$\overline{h} : C \to V$$

such that

(i) For x in B, $\overline{h}(x) = h(x)$;
(ii) For x, y in C,

$$\overline{h}(f(x, y)) = F(\overline{h}(x), \overline{h}(y)),$$
$$\overline{h}(g(x)) = G(\overline{h}(x)).$$

Viewed algebraically, the conclusion of this theorem says that any map h of B into V can be extended to a homomorphism $\overline{h}$ from C (with operations f and g) into V (with operations F and G).

If the content of the recursion theorem is not immediately apparent, try looking at it chromatically. You want to have a function $\overline{h}$ that paints each member of C some color. You have before you

1. h, telling you how to color the initial elements in B;
2. F, which tells you how to combine the color of x and y to obtain the color of $f(x, y)$ (i.e., it gives $\overline{h}(f(x, y))$ in terms of $\overline{h}(x)$ and $\overline{h}(y)$);

3. G, which similarly tells you how to convert the color of x into the color of $g(x)$.

The danger is that of a conflict in which, for example, F is saying "green" but G is saying "red" for the same point (unlucky enough to be equal both to $f(x, y)$ and $g(z)$ for some x, y, z). But if C is *freely* generated, then this danger is avoided.

Examples. Consider again the examples of the preceding subsection.

1. $B = \{0\}$ with one operation, the successor operation S. Then C is the set $\mathbb{N}$ of natural numbers. Since the successor operation is one-to-one and 0 is not in its range, C is freely generated from $\{0\}$ by S. Therefore, by the recursion theorem, for any set V, any $a \in V$, and any $F : V \to V$ there is a unique $\overline{h} : \mathbb{N} \to V$ such that $\overline{h}(0) = a$ and $\overline{h}(S(x)) = F(\overline{h}(x))$ for each $x \in \mathbb{N}$. For example, there is a unique $\overline{h} : \mathbb{N} \to \mathbb{N}$ such that $\overline{h}(0) = 0$ and $\overline{h}(S(x)) = 1 - \overline{h}(x)$. This function has the value 0 at even numbers and the value 1 at odd numbers.

2. The integers are generated from $\{0\}$ by the successor and predecessor operations but not freely generated.

3. Freeness fails also for the generation of the algebraic functions in the manner described.

4. The wffs are freely generated from the sentence symbols by the five formula-building operations. This fact is implicit in the parsing algorithm of the preceding section; we now want to focus on it here:

Unique Readability Theorem The five formula-building operations, when restricted to the set of wffs,

(a) Have ranges that are disjoint from each other and from the set of sentence symbols, and

(b) Are one-to-one.

In other words, the set of wffs is *freely* generated from the set of sentence symbols by the five operations.

Proof. To show that the restriction of $\mathcal{E}_\wedge$ is one-to-one, suppose that

$$(\alpha \wedge \beta) = (\gamma \wedge \delta),$$

where α, β, γ, and δ are wffs. Delete the first symbol of each sequence, obtaining

$$\alpha \wedge \beta) = \gamma \wedge \delta).$$

Then we must have $\alpha = \gamma$, lest one be a proper initial segment of the other (in contradiction to Lemma 13B). And then it follows at once that $\beta = \delta$. The same argument applies to $\mathcal{E}_\vee$, $\mathcal{E}_\rightarrow$, and $\mathcal{E}_\leftrightarrow$; for $\mathcal{E}_\neg$ a simpler argument suffices.

A similar line of reasoning tells us that the operations have disjoint ranges. For example, if

$$(\alpha \wedge \beta) = (\gamma \rightarrow \delta)$$

where α, β, γ, and δ are wffs, then as in the above paragraph we have $\alpha = \gamma$. But that implies that $\wedge = \rightarrow$, contradicting the fact that our symbols are distinct. Hence $\mathcal{E}_\wedge$ and $\mathcal{E}_\rightarrow$ (when restricted to wffs) have disjoint ranges. Similarly for any two binary connectives.

The remaining cases are simple. If $(\neg\,\alpha) = (\beta \wedge \gamma)$, then β begins with $\neg$, which no wff does. No sentence symbol is a sequence of symbols beginning with **(**. ⊣

Now let us return to the question of extending a truth assignment v to $\overline{v}$. First consider the special case where v is a truth assignment for the set of all sentence symbols. Then by applying the unique readability theorem and the recursion theorem we conclude that there is a unique extension $\overline{v}$ to the set of all wffs with the desired properties.

Next take the general case where v is a truth assignment for a set $\mathcal{S}$ of sentence symbols. The set $\overline{\mathcal{S}}$ generated from $\mathcal{S}$ by the five formula-building operations is freely generated, as a consequence of the unique readability theorem. So by the recursion theorem there is a unique extension $\overline{v}$ of v to that set, having the desired properties.

Example. We can apply the recursion theorem to establish that there is a unique function h defined on the set of wffs such that

$$\begin{aligned} \overline{h}(A) &= 1 \quad \text{for a sentence symbol} \quad A, \\ \overline{h}((\neg\,\alpha)) &= 3 + \overline{h}(\alpha), \\ \overline{h}((\alpha \wedge \beta)) &= 3 + \overline{h}(\alpha) + \overline{h}(\beta), \end{aligned}$$

and similarly for $\vee$, $\rightarrow$, and $\leftrightarrow$. This function gives the length of each wff.

Proof of the Recursion Theorem. The idea is to let $\overline{h}$ be the union of many approximating functions. Temporarily call a function v (which maps part of C into V) *acceptable* if it meets the conditions imposed on $\overline{h}$ by (i) and (ii). More precisely, v is acceptable iff the domain of v is a subset of C, the range a subset of V, and for any x and y in C:

(i′) If x belongs to B and to the domain of v, then $v(x) = h(x)$.

(ii′) If $f(x, y)$ belongs to the domain of v, then so do x and y, and $v(f(x, y)) = F(v(x), v(y))$. If $g(x)$ belongs to the domain of v, then so does x, and $v(g(x)) = G(v(x))$.

Let K be the collection of all acceptable functions, and let $\overline{h} = \bigcup K$, the union of all the acceptable functions. Thus

$$\begin{aligned}\langle x, z\rangle \in \overline{h} \quad &\text{iff} \quad \langle x, z\rangle \text{ belongs to some acceptable } v \\ &\text{iff} \quad v(x) = z \text{ for some acceptable } v.\end{aligned} \tag{1.1}$$

We claim that $\overline{h}$ meets our requirements. The argument is set-theoretic, and comprises four steps. First, here is an outline of four steps:

1. We claim that $\overline{h}$ is a function (i.e., that it is single-valued). Let

$$\begin{aligned}S &= \{x \in C \mid \text{ for at most one } z,\ \langle x, z\rangle \in \overline{h}\} \\ &= \{x \in C \mid \text{ all acceptable functions defined at } x \text{ agree there}\}\end{aligned}$$

It is easy to verify that S is inductive, by using (i′) and (ii′). Hence $S = C$ and $\overline{h}$ is a function.

2. We claim that $\overline{h} \in K$; i.e., that $\overline{h}$ itself is an acceptable function. This follows fairly easily from the definition of $\overline{h}$ and the fact that it is a function.

3. We claim that $\overline{h}$ is defined throughout C. It suffices to show that the domain of $\overline{h}$ is inductive. It is here that the assumption of freeness is used. For example, one case is the following: Suppose that x is in the domain of $\overline{h}$. Then $\overline{h}; \langle g(x), G(\overline{h}(x))\rangle$ is acceptable. (The freeness is required in showing that it is acceptable.) Consequently, $g(x)$ is in the domain of $\overline{h}$.

4. We claim that $\overline{h}$ is unique. For given two such functions, let S be the set on which they agree. Then S is inductive, and so equals C. ⊣

Now for the details.

1. As above, let

$$\begin{aligned}S &= \{x \in C \mid \text{ for at most one } z,\ \langle x, z\rangle \in \overline{h}\} \\ &= \{x \in C \mid \text{ all acceptable functions defined at } x \text{ agree there}\}\end{aligned}$$

Toward showing that S is inductive, first consider some x in B. Suppose that v_1 and v_2 are acceptable functions defined at x; we seek to show that $v_1(x) = v_2(x)$. But condition (i′) tells us that both $v_1(x)$ and $v_2(x)$ must equal $h(x)$, so indeed $v_1(x) = v_2(x)$. This shows that $x \in S$; since x was an arbitrary member of B we have $B \subseteq S$.

Secondly we must check that S is closed under f and g. So suppose that some x and y are in S; we ask whether $f(x, y)$ is in S. So suppose that v_1 and v_2 are acceptable functions defined at

$f(x, y)$; we seek to show that they agree there. But condition (ii′) tells us that $v_1(f(x, y)) = F(v_1(x), v_1(y))$ and $v_2(f(x, y)) = F(v_2(x), v_2(y))$. And because x and y are in S, we have $v_1(x) = v_2(x)$ and $v_1(y) = v_2(y)$ (and these are defined). So we conclude that $v_1(f(x, y)) = v_2(f(x, y))$. This shows that $f(x, y) \in S$. Hence S is closed under f. A similar argument shows that S is closed under g.

Thus S is inductive and so $S = C$. This shows that $\overline{h}$ is single-valued, i.e., is a function. Because $\overline{h}$ includes every acceptable function as a subset, we can say that

$$\overline{h}(x) = v(x)$$

whenever v is an acceptable function and $x \in \operatorname{dom} v$.

2. We claim that $\overline{h}$ is acceptable. Clearly $\operatorname{dom}\overline{h} \subseteq C$ and $\operatorname{ran}\overline{h} \subseteq V$ (by (∗)), and we have just verified that $\overline{h}$ is a function. It remains to check that $\overline{h}$ satisfies conditions (i′) and (ii′).

First we examine (i′). Assume $x \in B$ and $x \in \operatorname{dom}\overline{h}$ (so that $\langle x, \overline{h}(x)\rangle \in \overline{h}$). There must be some acceptable v with $v(x) = \overline{h}(x)$. Because v satisfies (i′), we have $v(x) = h(x)$ whence $\overline{h}(x) = h(x)$. So $\overline{h}$ satisfies (i′).

Secondly we examine (ii′). Assume that $f(x, y) \in \operatorname{dom}\overline{h}$. Again there must be some acceptable v with $v(f(x, y)) = \overline{h}(f(x, y))$. Because v satisfies (ii′), we have $v(f(x, y)) = F(v(x), v(y))$. Now $\overline{h}(x) = v(x)$ and $\overline{h}(y) = v(y)$ and hence

$$\overline{h}(f(x, y)) = v(f(x, y)) = F(v(x), v(y)) = F(\overline{h}(x), \overline{h}(y)).$$

In a similar way, we find that $\overline{h}(g(x)) = G(\overline{h}(x))$ whenever $g(x) \in \operatorname{dom}\overline{h}$. Hence $\overline{h}$ meets condition (ii′) and so is acceptable.

3. Next we must show that $\operatorname{dom}\overline{h}$ is inductive. First consider a point x in B. Then the set $\{\langle x, h(x)\rangle\}$ is a (small) acceptable function. For it clearly satisfies (i′). It also satisfies (ii′) because $x \notin \operatorname{ran} f_C$ and $x \notin \operatorname{ran} g_C$. Thus $\{\langle x, h(x)\rangle\}$ is acceptable and therefore is included in $\overline{h}$. Hence $x \in \operatorname{dom}\overline{h}$. This shows that $B \subseteq \operatorname{dom}\overline{h}$.

We further claim that $\operatorname{dom}\overline{h}$ is closed under f and g. Toward this end, consider any s and t in $\operatorname{dom}\overline{h}$. We hope that $f(s, t) \in \operatorname{dom}\overline{h}$. But if not, then let

$$v = \overline{h} \cup \{\langle f(s, t), F(\overline{h}(s), \overline{h}(t))\rangle\},$$

the result of adding this one additional pair to $\overline{h}$. It is clear that v is a function, $\operatorname{dom} v \subseteq C$, and $\operatorname{ran} v \subseteq V$. We claim that v satisfies (i′) and (ii′).

First take (i′). If $x \in B \cap \operatorname{dom} v$ then $x \neq f(s, t)$, by freeness. Hence $x \in \operatorname{dom} \overline{h}$ and we have $v(x) = \overline{h}(x) = h(x)$.

Next take (ii′). Assume that $f(x, y) \in \operatorname{dom} v$ for some x and y in C. If $f(x, y) \in \operatorname{dom} \overline{h}$ then $v(f(x, y)) = \overline{h}(f(x, y)) = F(\overline{h}(x), \overline{h}(y)) = F(v(x), v(y))$ since $\overline{h}$ is acceptable. The other possibility is that $f(x, y) = f(s, t)$. Then by freeness we have $x = s$ and $y = t$, and we know that these points are in $\operatorname{dom} \overline{h} \subseteq \operatorname{dom} v$. By construction,

$$\begin{aligned} v(f(s,t)) &= F(\overline{h}(s), \overline{h}(t)) \\ &= F(v(s), v(t)). \end{aligned}$$

Finally suppose that $g(x) \in \operatorname{dom} v$ for x in C. Then by freeness we have $g(x) \neq f(s, t)$. Hence $g(x) \in \operatorname{dom} \overline{h}$, and consequently $v(g(x)) = \overline{h}(g(x)) = G(\overline{h}(x)) = G(v(x))$.

Thus v is acceptable. But that tells us that $v \subseteq \overline{h}$, so that $f(s, t) \in \operatorname{dom} \overline{h}$ after all.

A similar argument shows that $\operatorname{dom} \overline{h}$ is closed under g as well. Hence $\operatorname{dom} \overline{h}$ is inductive and therefore coincides with C.

4. To show that $\overline{h}$ is unique, suppose that $\overline{h}_1$ and $\overline{h}_2$ both satisfy the conclusion of the theorem. Let S be the set on which they agree:

$$S = \{x \in C \mid \overline{h}_1(x) = \overline{h}_2(x)\}.$$

Then it is not hard to verify that S is inductive. Consequently $S = C$ and $\overline{h}_1 = \overline{h}_2$. ⊣

One final comment on induction and recursion: The induction principle we have stated is not the only one possible. It is entirely possible to give proofs by induction (and definitions by recursion) on the length of expressions, the number of places at which connective symbols occur, etc. Such methods are inherently less basic but may be necessary in some situations.

Exercises

1. Suppose that C is generated from a set $B = \{a, b\}$ by the binary operation f and unary operation g. List all the members of C_2. How many members might C_3 have? C_4?
2. Obviously $(\mathbf{A}_3 \rightarrow \wedge \mathbf{A}_4)$ is not a wff. But prove that it is not a wff.
3. We can generalize the discussion in this section by requiring of $\mathcal{F}$ only that it be a class of relations on U. $C_\star$ is defined as before, except that $\langle x_0, x_1, \ldots, x_n \rangle$ is now a construction sequence provided that for each $i \leq n$ we have either $x_i \in B$ or $\langle x_{j_1}, \ldots, x_{j_k}, x_i \rangle \in R$ for some $R \in \mathcal{F}$ and some $j_1, \ldots, j_k$ all less than i. Give the correct definition of $C^\star$ and show that $C^\star = C_\star$.