Cardinal Numbers

All infinite sets are big, but some are bigger than others. (For example, the set of real numbers is bigger than the set of integers.) Cardinal numbers provide a convenient, although not indispensable, way of talking about the size of sets.

It is natural to say that two sets A and B have the same size iff there is a function that maps A one-to-one onto B. If A and B are finite, then this concept is equivalent to the usual one: If you count the members of A and the members of B, then you get the same number both times. But it is applicable even to infinite sets A and B, where counting is difficult.

Formally, then, say that A and B are *equinumerous* (written $A \sim B$) iff there is a one-to-one function mapping A onto B. For example, the set $\mathbb{N}$ of natural numbers and the set $\mathbb{Z}$ of integers are equinumerous. It is easy to see that equinumerosity is reflexive, symmetric, and transitive.

For finite sets we can use natural numbers as measures of size. The same natural number would be assigned to two finite sets (as measures of their size) iff the sets were equinumerous. Cardinal numbers are introduced to enable us to generalize this situation to infinite sets.

To each set A we can assign a certain object, the *cardinal number* (or *cardinality*) of A (written card A), in such a way that two sets are assigned the same cardinality iff they are equinumerous:

$$\text{card } A = \text{card } B \quad \text{iff} \quad A \sim B. \tag{K}$$

There are several ways of accomplishing this; the standard one these days takes card A to be the least ordinal equinumerous with A. (The success of this definition relies on the axiom of choice.) We will not discuss ordinals here, since for our purposes it matters very little what card A actually is, any more than it matters what the number 2 actually is. What matters most is that (K) holds. It is helpful, however, if for a finite set A, card A is the natural number telling how many elements A has. Something is a *cardinal number*, or simply a *cardinal*, iff it is card A for some set A.

(Georg Cantor, who first introduced the concept of cardinal number, characterized in 1895 the cardinal number of a set M as "the general concept which, with the help of our active intelligence, comes from the set M upon abstraction from the nature of its various elements and from the order of their being given.")

Say that A is *dominated* by B (written $A \preceq B$) iff A is equinumerous with a subset of B. In other words, $A \preceq B$ iff there is a one-to-one function mapping A into B. The companion concept for cardinals is

$$\text{card } A \leq \text{card } B \quad \text{iff} \quad A \preceq B.$$

(It is easy to see that $\leq$ is well defined; that is, whether or not $\kappa \leq \lambda$ depends only on the cardinals κ and λ themselves, and not the choice of sets having these cardinalities.) Dominance is reflexive and transitive. A set A is dominated by $\mathbb{N}$ iff A is countable. The following is a standard result in this subject.

Schröder–Bernstein Theorem (a) For any sets A and B, if $A \preceq B$ and $B \preceq A$, then $A \sim B$.
(b) For any cardinal numbers κ and λ, if $\kappa \leq \lambda$ and $\lambda \leq \kappa$, then $\kappa = \lambda$.

Part (b) is a simple restatement of part (a) in terms of cardinal numbers. The following theorem, which happens to be equivalent to the axiom of choice, is stated in the same dual manner.

Theorem 0C (a) For any sets A and B, either $A \preceq B$ or $B \preceq A$.
(b) For any cardinal numbers κ and λ, either $\kappa \leq \lambda$ or $\lambda \leq \kappa$.

Thus of any two cardinals, one is smaller than the other. (In fact, any nonempty set of cardinal numbers contains a smallest member.) The smallest cardinals are those of finite sets: 0, 1, 2, ... There is next the smallest infinite cardinal, card $\mathbb{N}$, which is given the name $\aleph_0$. Thus we have

$$0, 1, 2, \ldots, \aleph_0, \aleph_1, \ldots,$$

where $\aleph_1$ is the smallest cardinal larger than $\aleph_0$. The cardinality of the real numbers, card $\mathbb{R}$, is called "$2^{\aleph_0}$." Since $\mathbb{R}$ is uncountable, we have $\aleph_0 < 2^{\aleph_0}$.

The operations of addition and multiplication, long familiar for finite cardinals, can be extended to all cardinals. To compute $\kappa + \lambda$ we choose disjoint sets A and B of cardinality κ and λ, respectively. Then

$$\kappa + \lambda = \operatorname{card}(A \cup B).$$

This is well defined; i.e., $\kappa + \lambda$ depends only on κ and λ, and not on the choice of the disjoint sets A and B. For multiplication we use

$$\kappa \cdot \lambda = \operatorname{card}(A \times B).$$

Clearly these definitions are correct for finite cardinals. The arithmetic of infinite cardinals is surprisingly simple (with the axiom of choice). The sum or product of two infinite cardinals is simply the larger of them:

Cardinal Arithmetic Theorem For cardinal numbers κ and λ, if $\kappa \leq \lambda$ and λ is infinite, then $\kappa + \lambda = \lambda$. Furthermore, if $\kappa \neq 0$, then $\kappa \cdot \lambda = \lambda$.

In particular, for infinite cardinals κ,

$$\aleph_0 \cdot \kappa = \kappa.$$

Theorem 0D For an infinite set A, the set $\bigcup_n A^{n+1}$ of all finite sequences of elements of A has cardinality equal to card A.

We already proved this for the case of a countable A (see Theorem 0B).

Proof. Each A^{n+1} has cardinality equal to card A, by the cardinal arithmetic theorem (applied n times). So we have the union of $\aleph_0$ sets of this size, yielding $\aleph_0 \cdot \text{card } A = \text{card } A$ points altogether. ⊣

Example. It follows that the set of algebraic numbers has cardinality $\aleph_0$. First, we can identify each polynomial (in one variable) over the integers with the sequence of its coefficients. Then by the theorem there are $\aleph_0$ polynomials. Each polynomial has a finite number of roots. To give an extravagant upper bound, note that even if each polynomial had $\aleph_0$ roots, we would then have $\aleph_0 \cdot \aleph_0 = \aleph_0$ algebraic numbers altogether. Since there are at least this many, we are done.

Since there are uncountably many (in fact, $2^{\aleph_0}$) real numbers, it follows that there are uncountably many (in fact, $2^{\aleph_0}$) transcendental numbers.