2.5.D. **Uniform substitution for schematic letters** In the logic of truth-functors the only schematic letters that we have are sentence-letters, so the principle concerns the substitution of arbitrary formulae in place of sentence-letters. It says that if we have any correct sequent, and if we substitute any formula for a sentence-letter in it—substituting the *same* formula for *every* occurrence of the sentence-letter, all through the sequent—then the result is again a correct sequent. It is useful to introduce a succinct notation for substitution. If $\phi$ and $\psi$ are formulae, and $P_i$ is a sentence-letter, we shall write $\phi(\psi/P_i)$ for the result of substituting an occurrence of the formula $\psi$ for each occurrence of the letter $P_i$ in $\phi$. (If there is no occurrence of $P_i$ in $\phi$, then $\phi(\psi/P_i)$ is just $\phi$.) Similarly, if $\Gamma$ is a set of formulae, then we shall write $\Gamma(\psi/P_i)$ for the result of substituting an occurrence of $\psi$ for each occurrence of $P_i$ throughout all the formulae in $\Gamma$. Then we may state our principle in two versions, corresponding to the two kinds of sequent we are recognizing:

(*a*) If $\Gamma \vDash \phi$ then $\Gamma(\psi/P_i) \vDash \phi(\psi/P_i)$.
(*b*) If $\Gamma \vDash$ then $\Gamma(\psi/P_i) \vDash$.

The justification for the principle is obvious at once. If we have a correct sequent containing a letter $P_i$, then that sequent satisfies the truth-table test whichever value is assigned to $P_i$. But when we replace $P_i$ by a different formula, still that formula as a whole can only take one of the values that $P_i$ could take, and therefore the truth-table test must still be satisfied. That means that the sequent is still correct.

Here are some simple illustrations. It is easily checked that the following is a correct entailment:

$$P \rightarrow \neg P \ \vDash\ \neg P.$$

We may therefore substitute any other formula for all the occurrences of $P$ in this entailment, and the result will again be an entailment; for example:

$$\begin{aligned} \bot \rightarrow \neg\bot &\models \neg\bot \\ Q \rightarrow \neg Q &\models \neg Q \\ \neg P \rightarrow \neg\neg P &\models \neg\neg P. \end{aligned}$$

These result by substituting for $P$ first $\bot$, then $Q$, then $\neg P$, which are very simple substitutions. But we may also substitute more complex formulae, say $P \wedge Q \wedge \neg R$, or $(P \rightarrow \neg P) \rightarrow \neg P$, to obtain

$$\begin{aligned} (P \wedge Q \wedge \neg R) \rightarrow \neg(P \wedge Q \wedge \neg R) &\models \neg(P \wedge Q \wedge \neg R) \\ ((P \rightarrow \neg P) \rightarrow \neg P) \rightarrow \neg((P \rightarrow \neg P) \rightarrow \neg P) &\models \neg((P \rightarrow \neg P) \rightarrow \neg P) \end{aligned}$$

To check the correctness of these last two sequents, it is a good deal easier to note that they are substitution-instances of a simple sequent already known to be correct, than it is to apply the truth-table test directly to them.

The question of what to count as 'basic' principles for the truth-functors will be taken up in more detail in Chapters 6 and 7. For the present, I set it aside, in order to come to an important principle which is naturally associated with the biconditional, though it is not at all the same in character as the principles proposed for the other truth-functors. In fact there are versions of it which do not rely on the biconditional at all, as we shall see.

**2.5.1. Interchange of equivalent formulae** The gist of this principle is that if two formulae are equivalent then either may be substituted for the other. In the strongest version of the principle, which I take first, formulae are taken to be equivalent if (in a given interpretation) they have the same truth-value. To state this more exactly, let $\phi$ and $\psi$ be any two formulae; let $\delta(\phi)$ be any formula which contains within itself one or more occurrences of the formula $\phi$ as a subformula; let $\delta(\psi)$ be the result of substituting the formula $\psi$ in place of the formula $\phi$, at one or more occurrences in $\delta(\phi)$. Then the principle in question is

$$\phi \leftrightarrow \psi \quad \vDash \quad \delta(\phi) \leftrightarrow \delta(\psi).$$

The proof is straightforward. The principle claims that any interpretation which verifies $\phi\leftrightarrow\psi$, and which also interprets $\delta(\phi)$, will verify $\delta(\phi) \leftrightarrow \delta(\psi)$. An interpretation which verifies $\phi\leftrightarrow\psi$ is one that assigns the same truth-value to both formulae. But then it must follow that that interpretation also assigns the same truth-value to $\delta(\phi)$ and $\delta(\psi)$. For $\delta(\phi)$ and $\delta(\psi)$ are exactly alike, except that the one has $\phi$ in some places where the other has $\psi$. But if $\phi$ and $\psi$ have the same value, then this difference will not affect the calculation of the values of $\delta(\phi)$ and $\delta(\psi)$.

This is the basic form of the principle of interchange of equivalent formulae. In practice, the principle is often used in a weaker form, which confines attention to formulae which are *logically* equivalent, i.e. which have the same truth-value in *all* interpretations. In this form the principle is

$$\text{If } \vDash \phi \leftrightarrow \psi \text{ then } \vDash \delta(\phi) \leftrightarrow \delta(\psi).$$

It is clear that this follows (by the principle of Cutting) from the first version. We may rephrase this derived form in a way which eliminates the truth-functor $\leftrightarrow$. For as a special case of our basic principle for the conditional we have

$$\vDash \phi\rightarrow\psi \quad \text{iff} \quad \phi \vDash \psi$$

and hence also

$$\vDash \phi\leftrightarrow\psi \quad \text{iff} \quad \phi \vDash \psi \text{ and } \psi \vDash \phi.$$

Abbreviating the right-hand side of this to '$\phi \dashv\vDash \psi$', we may therefore write the derived form of the principle in this way:

$$\text{If } \phi \dashv\vDash \psi \text{ then } \delta(\phi) \dashv\vDash \delta(\psi).$$

There will be several applications of this form of the principle in what follows.

We are now in a position to move on to the two remaining principles to be introduced in this section, namely the principle of uniform substitution for schematic letters, and the principle of interchange of equivalent formulae. These principles hold for our languages with quantifiers just as they did for our languages for truth-functors, but they are now very much more complicated to state and to justify. I begin with the interchange of equivalent formulae.

In the languages for truth-functors of the last chapter, there was no distinction to be made between open and closed formulae, for all formulae were closed. Consequently, the principle allowing for interchange of equivalent formulae was there confined to closed formulae, which makes it very simple to state and to prove. But now we have open formulae to consider as well, for they too can be equivalent to one another, and if so then they too can be interchanged while preserving the equivalence of the whole. If $\phi$ and $\psi$ are open formulae, then they are interpreted as equivalent iff the universal closure of the biconditional formed from them is interpreted as true. Thus if the free variables in $\phi$ are just $x$ and $y$, and the same holds for $\psi$, then to say that $\phi$ and $\psi$ are equivalent in an interpretation $I$ is just to say that $\forall x \forall y(\phi \leftrightarrow \psi)$ is true in $I$ (and to say that $\phi$ and $\psi$ are *logically* equivalent is to say $\forall x \forall y(\phi \leftrightarrow \psi)$ is true in all interpretations, i.e. is valid). More generally, where $\phi$ and $\psi$ are any formulae, with any number of free variables $\xi_1 ... \xi_n$, I shall write

$$\forall \xi_1 ... \xi_n(\phi \leftrightarrow \psi)$$

to signify the closure of their biconditional. (If $\phi$ and $\psi$ are both *closed* formulae, then $n$=0.)

Now, equivalent formulae may be substituted for one another, preserving truth-value. To state this succinctly, let $\phi$ and $\psi$ be any formulae, whether closed or open, and let $\delta(\phi)$ be any closed formula containing $\phi$ as a subformula, and $\delta(\psi)$ be the result of interchanging $\phi$ and $\psi$ at one or more places in $\delta(\phi)$. Then the basic principle that we require can be stated in this way:

**3.6.H.** $\forall\xi_1...\xi_n(\phi\leftrightarrow\psi) \vDash \delta(\phi) \leftrightarrow \delta(\psi)$.

If we had adopted the semantics on pp. 86–9 based on the notion of satisfaction, then it would at once be obvious that this entailment is correct. For if two formulae are equivalent in a certain interpretation, then it is easy to see that they must be satisfied by all the same assignments in that interpretation. Consequently, they must make the same contribution to the values of any longer formula that contains them. (That is the analogue of the justification given on p. 32, for the simple version of this principle that applies in quantifier-free languages.) But as things are, the semantics that we have adopted assigns no values to open formulae, so our justification must be more roundabout. For the sake of later developments (Exercise 6.2.2) I shall here give an argument by induction, namely an induction on the number of occurrences of truth-functors and quantifiers that are in $\delta(\phi)$ but not in $\phi$.

In fact, it turns out to be convenient to prove slightly more than 3.6.H as just formulated. Let $\phi'$ be any formula resulting from $\phi$ by substituting name-letters for zero or more of the variables free in $\phi$ (substituting the same name-letter for each occurrence of the same variable), and let $\psi'$ result from $\psi$ by the same substitutions. Then what we shall prove is

$$\forall\xi_1...\xi_n(\phi\leftrightarrow\psi) \vDash \delta(\phi') \leftrightarrow \delta(\psi').$$

The hypothesis of induction is

If $\theta(\phi')$ is shorter than $\delta(\phi')$ then
$\forall\xi_1...\xi_n(\phi\leftrightarrow\psi) \vDash \theta(\phi') \leftrightarrow \theta(\psi')$.

We shall again suppose that the language we are concerned with contains only $\neg,\wedge,\forall$ as its logical vocabulary, so that we have four cases to consider.

*Case (1)*: $\delta(\phi')$ is no longer than $\phi'$, i.e. $\delta(\phi')$ is $\phi'$. Then since $\delta(\phi')$ is closed (by hypothesis), $\phi'$ is closed, and therefore it must result from $\phi$ by substituting name letters for the variables (if any) that are free in $\phi$. So the entailment to be established in this case is

$$\forall\xi_1...\xi_n(\phi\leftrightarrow\psi) \vDash (\phi\leftrightarrow\psi)(\alpha_1/\xi_1, \alpha_2/\xi_2,...,\alpha_n/\xi_n).$$

But this is obviously a correct entailment, as may be shown by repeated use of the principle of $\forall$-elimination.

*Case (2)*: $\delta(\phi')$ is $\neg\theta(\phi')$. Then by inductive hypothesis we have

$$\forall\xi_1...\xi_n(\phi\leftrightarrow\psi) \models \theta(\phi') \leftrightarrow \theta(\psi')$$

and by a simple truth-functional inference we have

$$\theta(\phi') \leftrightarrow \theta(\psi') \models \neg\theta(\phi') \leftrightarrow \neg\theta(\psi').$$

From these two the result evidently follows, by CUT.

*Case (3)*: $\delta(\phi')$ is $\theta_1(\phi') \wedge \theta_2(\phi')$, where $\phi'$ may perhaps be missing from one of the conjuncts. (This will not affect the argument.) Then by inductive hypothesis we have

$$\forall\xi_1...\xi_n(\phi\leftrightarrow\psi) \models \theta_1(\phi') \leftrightarrow \theta_1(\psi')$$
$$\forall\xi_1...\xi_n(\phi\leftrightarrow\psi) \models \theta_2(\phi') \leftrightarrow \theta_2(\psi').$$

From this the result follows by a simple truth-functional inference, as in case (2).

*Case (4)*: $\delta(\phi')$ is $\forall\zeta\theta(\phi')$. Let $\beta$ be a new name, not occurring in $\theta(\phi')$ or $\theta(\psi')$. (Note that it follows that $\beta$ does not occur in $\phi$ or in $\psi$.) Then by inductive hypothesis we have

$$\forall\xi_1...\xi_n(\phi\leftrightarrow\psi) \models (\theta(\phi'))(\beta/\zeta) \leftrightarrow (\theta(\psi'))(\beta/\zeta).$$

That is

$$\forall\xi_1...\xi_n(\phi\leftrightarrow\psi) \models (\theta(\phi') \leftrightarrow \theta(\psi'))(\beta/\zeta).$$

Since $\beta$ does not occur in the premiss, we may apply $\forall$-introduction to this to obtain

$$\forall\xi_1...\xi_n(\phi\leftrightarrow\psi) \models \forall\zeta(\theta(\phi') \leftrightarrow \theta(\psi'))(\beta/\zeta)(\zeta/\beta).$$

But since $\beta$ does not occur in $(\theta(\phi') \leftrightarrow \theta(\psi'))$, this is just

$$\forall\xi_1...\xi_n(\phi\leftrightarrow\psi) \models \forall\zeta(\theta(\phi') \leftrightarrow \theta(\psi')).$$

And we have already proved as a lemma (pp. 99–100)

$$\forall\zeta(\theta(\phi') \leftrightarrow \theta(\psi')) \models \forall\zeta\theta(\phi') \leftrightarrow \forall\zeta\theta(\psi').$$

So the desired result now follows by CUT.

This completes the induction.

I remark that in the statement of this principle we have required $\delta(\phi')$ and $\delta(\psi')$ to be closed, as this simplifies the argument above. But we could allow them to be open formulae, with free variables $\zeta_1...\zeta_m$, and in this case the correct statement of the principle will be

**3.6.H.**(*a*) $\forall\xi_1...\xi_n(\phi\leftrightarrow\psi) \models \forall\zeta_1...\zeta_m(\delta(\phi') \leftrightarrow \delta(\psi'))$.

It is easy to see how this version can be established from what we have already. For if in $(\delta(\phi') \leftrightarrow \delta(\psi'))$ we write new name-letters in place of the free variables, then we have an entailment that is an instance of 3.6.H, as already established. But then by repeated applications of ∀-introduction these name-letters can be restored to variables, and universally quantified, as the new version requires.

As before, there is a weaker version of this principle, stating that *logically* equivalent formulae may be interchanged, i.e.

3.6.H.(*b*) If $\models \forall\xi_1...\xi_n(\phi\leftrightarrow\psi)$ then $\models \forall\zeta_1...\zeta_m(\delta(\phi') \leftrightarrow \delta(\psi'))$.

From this weaker version (which in practice is more often useful) we can again eliminate the functor ↔ if we wish, but I leave that as an exercise.

It is worth mentioning one simple corollary of his principle of interchange, namely that we may always in roduce an *alphabetic change of bound variable.* Consider first any formula that begins with a quantifier, say $Q\xi\phi(\xi)$, where Q is either ∀ or ∃. Let $\phi(\zeta)$ be the result of substituting occurrences of the different variable $\zeta$ for all free occurrences of $\xi$ in $\phi(\xi)$, assuming that the substituted occurrences re free in $\phi(\zeta)$, and that $\phi(\xi)$ does not already contain any free occurrences of $\zeta$. Thus $\phi(\xi)$ contains $\xi$ free wherever and only where $\phi(\zeta)$ contains $\zeta$ free. In that case it is easy to see that

$$Q\xi\phi(\xi) \dashv\models Q\zeta\phi(\zeta),$$

for the truth-conditions for each form la are exactly the same. B the principle of interchange, then, we have

$$\delta(Q\xi\phi(\xi)) \dashv\models \delta(Q\zeta\phi(\zeta))$$

for any added matter δ. And, as we have seen, the result can also be generalized to cover the case where $Q\xi\phi(\xi)$, and hence $Q\zeta\phi(\zeta)$, are open formulae, containing other variables free. So we may say that, in any context whatever, one bound variable may always be exchanged for another (by 'relettering'), so long as the same bondage links are preserved. This operation of relettering is quite often useful, as we shall see in the next section.

Our final principle in this section is that which permits uniform substitution for schematic letters throughout a sequent. In the logic of truth-functors we had only one kind of schematic letter to consider namely the sentence-letters, and so again the principle was simple to state and to prove. We now have two kinds of schematic letters, i.e. name-letters and predicate-letters, and the principle of substitution holds for both of these. So we must take it in two parts. I consider first substitution for name-letters, since this is very straightforward. For the only expression that we have, that is of the same syntactical category as a name-letter, is another name-letter; and thus the principle simply permits us to substitute one name-letter for another. (The position will become a little more interesting in Chapter 8, where complex name-symbols will be introduced.) But substitution for a predicate-letter is a more complex operation, as we shall see.

I write $\Gamma(\beta/\alpha)$ for the result of substituting the name-letter $\beta$ for all occurrences of the name-letter $\alpha$ throughout all the formulae in $\Gamma$, and $\phi(\beta/\alpha)$ for the result of making the same substitution in the formula $\phi$. We can now state the principle required in two versions, according to the two kinds of sequent that we are recognizing.

**3.6.I.**(*a*) Uniform substitution for name-letters.

(1) If $\Gamma \vDash \phi$ then $\Gamma(\beta/\alpha) \vDash \phi(\beta/\alpha)$.
(2) If $\Gamma \vDash$ then $\Gamma(\beta/\alpha) \vDash$.

I sketch a proof just for the second case.

Assume that $\Gamma(\beta/\alpha) \nvDash$. That is, there exists an interpretation $I$ such that, for all formulae $\psi$ in $\Gamma$, $|\psi(\beta/\alpha)|_I = T$. Let $I_\alpha$ be an $\alpha$-variant of $I$, agreeing with $I$ in all respects, except that $I_\alpha$ interprets $\alpha$ as having the same denotation as does $\beta$ in $I$. Now $\alpha$ does not occur in $\psi(\beta/\alpha)$, and hence by the lemma on interpretations we have $|\psi(\beta/\alpha)|_{I_\alpha} = |\psi(\beta/\alpha)|_I$. Moreover, $|\alpha|_{I_\alpha} = |\beta|_{I_\alpha}$, and the formulae $\psi$ and $\psi(\beta/\alpha)$ result from one another by interchanging $\alpha$ and $\beta$ at suitable places. Hence by the lemma on extensionality $|\psi(\beta/\alpha)|_{I_\alpha} = |\psi|_{I_\alpha}$. Putting these equations together, $|\psi|_{I_\alpha} = T$. That is to say: there is an interpretation, namely $I_\alpha$, such that, for all formulae $\psi$ in $\Gamma$, $|\psi|_{I_\alpha} = T$. In other words, $\Gamma \nvDash$.

This argument shows: if $\Gamma(\beta/\alpha) \nvDash$, then $\Gamma \nvDash$. Contraposing, we have our result.

Turning to substitution for predicate-letters, we must first pause to explain what the relevant operation is. An $n$-place predicate-letter is immediately followed by a series of $n$ terms (either name-letters or variables), but it may be followed by different terms at different occurrences in the same formula. When we substitute for the predicate-letter, we are not at the same time substituting for the terms that follow it, so they must be preserved (in the right order) even though the predicate-letter is replaced by something else. Of course, this presents no problem if the predicate-letter is simply replaced by another predicate-letter, but in fact we have more interesting substitutions to consider. We said earlier (pp. 74–5) that an open sentence, with $n$

free variables, represents an $n$-place predicate. Similarly, an open *formula*, with $n$-free variables, is a complex schematic expression for complex $n$-place predicates of a certain structure. For example, the open formula

$$Fxy \land \neg Fyx$$

represents such complex predicates as can be obtained by substituting genuine open sentences in place of its atomic parts, as in

$x$ loves $y \land \neg y$ loves $x$
$x$ married $y \land \neg y$ married $x$
$x$ weighs more than $y \land \neg y$ weighs more than $x$.
etc.

Clearly, what holds for *all* two-place predicates also holds for all two-place predicates of this particular structure. That is to say that if we have a correct sequent, which holds no matter what two-place predicate a letter $G$ is taken to be, and if we substitute for that letter $G$ the open formula $Fxy \land \neg Fyx$, then the result must again be a correct sequent. In a word, the substitutions to be considered are these: for a zero-place predicate-letter (i.e. a sentence-letter), we may substitute any formula with zero free variables (i.e. any closed formula); and for an $n$-place predicate-letter ($n>0$), we may substitute any open formula with $n$ free variables. In the course of substituting an open formula for a predicate-letter, the free variables of that formula will disappear, to be replaced by the terms immediately following the predicate-letter on that occurrence. More precisely, the free variables of the open formula must be ordered in some way, say alphabetically, and we shall let this ordering correspond to the natural ordering of the terms immediately following an occurrence of the predicate-letter, namely from left to right. Then, each occurrence of the predicate-letter is replaced by the open sentence in question, and the alphabetically first free variable of the open sentence is replaced by the first from the left of the terms following the predicate letter at that occurrence, the second by the second, and so on. Here is an example. Suppose we begin with a sequent which claims (correctly) that an asymmetrical relation must be irreflexive:

$$\forall x \forall y (Fxy \rightarrow \neg Fyx) \models \forall x \neg Fxx.$$

For the schematic letter $F$ in this sequent we then substitute the open sentence

$$\exists z (Fxz \land Fzy).$$

The result is

$$\forall x\forall y(\exists z(Fxz \wedge Fzy) \rightarrow \neg\exists z(Fyz \wedge Fzx)) \models \forall x\neg\exists z(Fxz \wedge Fzx).$$

Since the original sequent was in fact a correct sequent, so too is this one obtained from it by substitution.

There is one caveat that needs to be entered. Variables immediately following a predicate-letter are, of course, free in the atomic formula so formed. When an open sentence is substituted for the predicate-letter, and the variables following the predicate-letter are substituted into that open sentence at appropriate positions, *they must remain free* in the open sentence so formed. If the result of a substitution would be that some previously free variables become bound by quantifiers in the open sentence, then the substitution cannot be performed. For example, in the sequent

$$\forall x\forall y(Fxy \rightarrow \neg Fyx) \models \forall x\neg Fxx$$

one cannot substitute for the schematic letter the open formula

$$\exists y(Fxy \wedge Fyz).$$

The result could only be

$$\forall x\forall y(\exists y(Fxy \wedge Fyy) \rightarrow \neg\exists y(Fyy \wedge Fyx)) \models \forall x\neg\exists y(Fxy \wedge Fyx).$$

But in this first formula the two atomic subformulae $Fyy$ each contain an occurrence of $y$ that *should* be bound by the initial quantifier $\forall y$, if the overall structure of the formula is to be preserved, whereas it has instead got captured by the nearer occurrence of the quantifier $\exists y$. This is illegitimate, and there is no way of substituting just that open formula for the schematic letter in that particular context. (Instead, one must first 'reletter' the bound variables of the open formula.)

To have a succinct notation, let us write $\Phi^n$ for an $n$-place predicate-letter, $\phi^n$ for a formula with $n$ variables free, and $\psi(\phi^n/\Phi^n)$ for the result of substituting the formula for the letter, according to the method just given, throughout the formula $\psi$. We assume that the substitution is a legitimate one. Similarly, we may write $\Gamma(\phi^n/\Phi^n)$ for the result of making such a substitution in every formula in $\Gamma$. Then our principle may again be stated in two versions:

**3.6.I.**(*b*) Uniform substitution for predicate-letters.

(1) If $\Gamma \models \psi$ then $\Gamma(\phi^n/\Phi^n) \models \psi(\phi^n/\Phi^n)$.
(2) If $\Gamma \models$ then $\Gamma(\phi^n/\Phi^n) \models$.

I give a proof just for the second version.

The proof will make the simplifying assumption that the letter $\Phi^n$ does not occur in the formula $\phi^n$ that is substituted for it. This restricted form of the principle in fact implies its unrestricted form, as we may see in this way. Let $\Psi^n$ be a new $n$-place predicate-letter, different from $\Phi^n$ and not occurring in $\phi^n$ or in any formula in $\Gamma$. Then by the restricted principle we may first substitute $\Psi^n$ for $\Phi^n$ throughout all the formulae in $\Gamma$, and next substitute $\phi^n$ for $\Psi^n$. The result of these two steps of substitution is just the same as the result of substituting $\phi^n$ for $\Phi^n$ directly. Let us come now to the proof.

Assume that $\Gamma(\phi^n/\Phi^n) \not\models$, i.e. that there is an interpretation $I$ such that, for all formulae $\psi$ in $\Gamma$, $|\psi(\phi^n/\Phi^n)|_I = T$. Now the interpretation $I$ must assign some extension to the open formula $\phi^n$, i.e. a set of $n$-tuples from the domain which $\phi^n$ may be counted as true of. Let us suppose that the free variables of $\phi^n$, in alphabetical order, are $x_1,...,x_n$. Then in our second method of defining an interpretation (pp. 86–9) the relevant $n$-tuples are just the $n$-tuples $\langle s(x_1),...,s(x_n)\rangle$ for those assignments s that satisfy $\phi^n$ in $I$. Alternatively, if we retain our first way of defining interpretations, we must first substitute new names $a_1,...,a_n$, not already occurring in $\phi^n$, for its free variables $x_1,...,x_n$, thus forming the closed formula $\phi^*$. Then the $n$-tuples that we require are just those $n$-tuples $\langle|a_1|,...,|a_n|\rangle$ formed from the denotations of these names in all interpretations which (1) agree with $I$ on all symbols other than the names $a_1,...,a_n$, and (2) make $\phi^*$ true. We can therefore introduce a new interpretation $\mathcal{J}$, which agrees with $I$ in all respects except that it assigns this set of $n$-tuples to the predicate-letter $\Phi^n$ as its extension. Since we are assuming that $\Phi^n$ does not occur in $\phi^n$, this leaves unchanged the interpretation of all symbols in $\Gamma(\phi^n/\Phi^n)$, so that we have $|\psi(\phi^n/\Phi^n)|_{\mathcal{J}} = T$, for all $\psi$ in $\Gamma$. But also, we have constructed $\mathcal{J}$ so that $\phi^n$ and $\Phi^n$ have the same extension in it, i.e. they are equivalent formulae. That is, we have as true in $\mathcal{J}$

$$\forall x_1...x_n(\phi^n \leftrightarrow \Phi^n).$$

Moreover, $\psi$ differs from $\psi(\phi^n/\Phi^n)$ just by having $\Phi^n$ at some places where the other has $\phi^n$. So, by our principle for interchanging equivalent formulae, it follows that $\psi$ and $\psi(\phi^n/\Phi^n)$ are equivalent in $\mathcal{J}$. Hence $|\psi|_{\mathcal{J}} = T$ for all $\psi$ in $\Gamma$, and therefore $\Gamma \not\models$, as desired. This completes the proof.