2.5. **Some Principles of Entailment**

It is useful to be familiar with a number of general principles concerning entailment (or inconsistency). We may first note three that are often called 'structural' principles, since they apply to formulae of any kind whatever, and not just to formulae of the languages for truth-functors that we are presently concerned with. They are called the principles of *Assumptions*, of *Thinning*, and of *Cutting*.

2.5.A. **Assumptions**

This is the principle that any formula entails itself, i.e.

$$\phi \vDash \phi.$$

(The reason why it is called the principle of assumptions will emerge in Chapters 6 and 7.) When we bear in mind the definition of entailment in terms of truth and falsehood in an interpretation, we see that this principle depends just upon the fact that no interpretation assigns *both* T *and* F to the same formula. It should be obvious enough that this is a fact, at least for (standard) interpretations of a language of truth-functors. You might like to reflect upon how it could be proved. (I give a proof in Section 2.8 below.)

2.5.B. **Thinning**

This is the principle that if a set of premisses entails a conclusion, and we *add* further premisses to that set, then the enlarged set still entails the conclusion. We have two versions of this principle to record, first for the ordinary case where our sequent has a conclusion, and second for the special case where there is no conclusion

(*a*) If $\Gamma \vDash \phi$ then $\Gamma, \psi \vDash \phi$.
(*b*) If $\Gamma \vDash$ then $\Gamma, \psi \vDash$.

(The principle is called 'Thinning' simply because thinning is a way of weakening, and '$\Gamma, \psi \vDash \phi$' makes a weaker claim than does '$\Gamma \vDash \phi$'.) In both these versions the principle allows us, if we wish, to add an extra formula to the *left* of the turnstile, so we may distinguish this as *Thinning on the left*.

Later, when we come to consider sequents with more than one formula on the right (in Chapter 7), a precisely analogous principle will allow us to add an extra formula to the right of the turnstile, and this is *Thinning on the right.* One special case of this can be stated now, namely where the number of formulae on the right is increased from zero to one:

(*c*) If $\Gamma \models$ then $\Gamma \models \psi$.

One has only to consider the definition of the turnstile, in these various contexts, and it is at once obvious that each of (*a*), (*b*), and (*c*) is a correct principle.

2.5.C. Cutting This principle is a generalization of the point that entailment is transitive, i.e. that if one formula entails a second, and the second entails a third, then the first formula entails the third. The generalization extends this to cover also entailments which have more than one premiss. Again, we have two versions to record, one where our 'third formula' is indeed a formula, and one where it is instead the absence of any formula:

(*a*) If $\Gamma \models \phi$ and $\phi,\Delta \models \psi$ then $\Gamma,\Delta \models \psi$
(*b*) If $\Gamma \models \phi$ and $\phi,\Delta \models$ then $\Gamma,\Delta \models$.

(It is called the principle of Cutting because the intermediate conclusion ϕ is 'cut out'.) This principle is not quite so obvious as the preceding two, so I here give a proof of version (*a*). (The modification to yield version (*b*) is obvious.)

Assume, for *reductio ad absurdum*, that the principle is not correct, i.e. that (for some Γ,Δ,ϕ,ψ) we have[1]

(1) $\Gamma \models \phi$ (2) $\phi,\Delta \models \psi$ (3) $\Gamma,\Delta \not\models \psi$.

Then by assumption (3) there is an interpretation *I* which assigns T to each formula in Γ, and to each in Δ, but assigns F to ψ. We ask: what value does *I* assign to ϕ? It may be that *I* assigns no value to ϕ, but if so that can only be because ϕ contains vocabulary which does not occur in Γ or Δ or ψ, and is not interpreted by *I*. In that case, we can evidently expand the interpretation *I*, by adding to it interpretations of the extra vocabulary of ϕ, to form a new interpretation I^+. Since I^+ agrees with *I* on the interpretation of all the vocabulary in Γ and Δ and ψ, it will still be the case that I^+ assigns T to all formulae in Γ, and T to all formulae in Δ, and F to ψ. But I^+ now does assign

[1] Recall that '$\not\models$' negates '$\models$'. So (3) means: not ($\Gamma,\Delta \models \psi$).

some value, either T or F, to ϕ. However, by assumption (1) $\Gamma \models \phi$, so I^+ cannot assign F to ϕ (since it assigns T to all in Γ); and by assumption (2) $\phi,\Delta \models \psi$, so I^+ cannot assign T to ϕ (since it assigns T to all in Δ but F to ψ). This is a contradiction. It follows, then, that assumptions (1), (2), and (3) cannot all be true, so that if (1) and (2) are true, then (3) must be false, as desired.

3.6. **Some Principles of Entailment**

It is easy to see that the so-called 'structural' principles of pp. 30–2 apply to our languages for quantifiers just as well as to our languages for truth-functors. These were

3.6.A. The principle of Assumptions (ASS)
3.6.B. The principle of Thinning (THIN)
3.6.C. The principle of Cutting (CUT).

Nothing more needs to be said about the proofs of the first two, which are the same as before in each case, but it is useful to add something here about the third.

If you look back to the proof of CUT given on pp. 31–2, you will see that it relies on this assumption:

> An interpretation I which interprets a set of formulae Γ, but does not interpret a formula ϕ, can always be expanded to an interpretation I^+ which assigns the same values to the formulae in Γ and assigns some value to ϕ as well.

The assumption would not have been correct if we had allowed an interpretation to have an empty domain of discourse. For, as I have noted (p. 85), if the formulae in Γ contain no name-letters, then they can all be interpreted on an empty domain, whereas if ϕ does contain a name-letter, then it cannot be. But changing the domain from an empty one to a non-empty one may well disturb the values assigned to the formulae in Γ. (For example, the two formulae $\exists xFx$ and $\exists x \neg Fx$ can both be false only if the domain is empty.) As things are, however, we are not permitting a domain to be empty, so every formula can be interpreted on every domain, and this obstacle is avoided. It then follows from our lemma 3.5.A on interpretations that the assumption just cited is satisfied by our semantics for quantifiers, and CUT can therefore be proved in the same way as before.