6

Natural Deduction

6.1. **The Idea** 239
6.2. **Rules of Proof I: Truth-Functors** 242
6.3. **Rules of Proof II: Quantifiers** 254
6.4. **Alternative Styles of Proof** 262
6.5. **Interim Review** 269

6.1. The Idea

Axiomatic proofs are hard to construct, and often very lengthy. So in practice one does not actually construct such proofs; rather, one proves that *there is* a proof, as originally defined. One way in which we make use of this technique is when we allow ourselves to use, in a proof, any theorem that has been proved already. For officially this is short for writing out once more, as part of the new proof, the whole of the original proof of that theorem. Another way is when we are explicitly relying on the deduction theorem, and so are actually concerned with a proof from assumptions, and not an axiomatic proof as first defined. Proofs from assumptions are much easier to find, and much shorter. A third way is when we introduce new symbols by definition, for in practice one will go on at once to derive new rules for the new symbols, and these will usually be rules for use in proofs from assumptions. So it comes about that, after a few initial moves, the development of an axiomatic system will scarcely ever involve writing out real axiomatic proofs, but will rely on a number of short cuts.

The main idea behind what is called 'natural deduction' is to abandon the axiomatic starting-point altogether, and instead to *begin* with what I have just been calling the 'short cuts'. The most important point is that in natural deduction one takes the notion of a proof from assumptions as a basic notion, and works simply with it. Such proofs are not thought of as abbreviating some other and more basic kind of proofs, but are the primary objects of study. So from the beginning our basic rules will be rules for use in proofs from assumptions, and axioms (as traditionally understood) will have no role to play. That is the most crucial feature of all systems of natural deduction. But there are several other features too that are nowadays expected and desired.

First, the truth-functor $\rightarrow$ will no longer have any special prominence. In axiomatic treatments it almost always does, both because the main rule of inference, namely detachment, is a rule for $\rightarrow$, and because there are only a very few formulae that one might naturally think of adopting as axioms and that do not have $\rightarrow$ (or $\leftrightarrow$) as their main functor. (The only obvious exceptions are the laws of excluded middle and non-contradiction, i.e. $\vdash \varphi\vee\neg\varphi$ and $\vdash \neg(\varphi\wedge\neg\varphi)$.) But we shall now have no axioms, and put no special weight on detachment (or Modus Ponens). Instead, we shall have separate rules for each truth-functor of the language to be employed, so that there will not only be rules for $\rightarrow$, but also for $\neg,\wedge,\vee$, and any other functor that is desired. To illustrate, a very natural principle for $\wedge$ is this: given both φ and ψ as premisses, one may infer $\varphi\wedge\psi$. If we try to phrase this as an axiom, then probably the simplest way is this:

$$\vdash \varphi\rightarrow(\psi\rightarrow\varphi\wedge\psi).$$

Here, of course, we use $\rightarrow$ as well as $\wedge$. But evidently the principle can also be formulated as a rule of inference which does not use $\rightarrow$. As a rule for use in axiomatic systems it would be

$$\text{If } \vdash \varphi \text{ and } \vdash \psi \text{ then } \vdash \varphi\wedge\psi.$$

(In this form it is called 'the rule of adjunction'.) But for use in proofs from assumptions we shall adopt the more general version

$$\text{If } \Gamma\vdash \varphi \text{ and } \Delta\vdash \psi \text{ then } \Gamma,\Delta\vdash \varphi\wedge\psi.$$

Given the structural rules ASS and CUT in the background, it is easy to show that this is actually equivalent to the simpler version

$$\varphi,\psi\vdash \varphi\wedge\psi.$$

Let us come back to the task of giving a general characterization of what is nowadays called 'natural deduction'. I have said so far (1) that the basic notion is that of a proof from assumptions, (2) that there will accordingly be no axioms (as traditionally understood) but a number of rules of inference for use in such proofs, and (3) that we shall expect to find, for each truth-functor or quantifier in the language being considered, rules that specifically concern it, and no other truth-functor or quantifier. Now (3) is more a requirement of elegance than a condition on what can be counted as natural deduction, and certainly systems have been proposed which one would wish to call systems of natural deduction even though they do not entirely conform to it. The same applies to this further elaboration of (3): for each truth-functor or quantifier concerned, there will be one or two rules that are counted as its *introduction* rules, and one or two that are counted as its *elimination* rules, and no other rules. Again, there are well-known systems which do not entirely conform to this, but it is what one expects nowadays. We can illustrate by continuing with our example of the functor $\wedge$. This has just one introduction rule, henceforward called ($\wedge$I), namely the rule already stated

($\wedge$I) $\varphi,\psi \vdash \varphi\wedge\psi$.

It has a pair of elimination rules, each (for brevity) called ($\wedge$E), namely

($\wedge$E) $\varphi\wedge\psi \vdash \varphi, \quad \varphi\wedge\psi \vdash \psi$.

And there are no other rules for $\wedge$. Moreover, we may add here a fourth requirement on systems of natural deduction, which is certainly a requirement of elegance and nothing more, for in fact I know of no system which succeeds in conforming to it without exception. This is (4)(*a*) that the introduction and elimination rules for any one sign be complete for that sign, in the sense that all correct sequents involving only that sign be provable from those rules alone; and (*b*) that combining the introduction and elimination rules for any two or more signs yields a system complete for those signs together, again in the sense that all correct sequents containing only those signs be provable from those rules alone.

Finally, I add two more requirements, of which it is evident that there is no fully objective way of telling whether they are satisfied or not. These are: (5) that the rules for each sign be 'natural', in the sense that inferences drawn in accordance with them strike us as 'natural' ways of arguing and inferring; and (6) that so long as the sequent that we are trying to prove is 'not too complicated', there should be a proof of it which is 'reasonably short' and uses only the rules initially adopted. As we observed earlier, in an axiomatic system it is necessary in practice to proceed in a cumulative fashion: after a brief initial development, one's proofs seldom go back to the original axioms, but rely instead on other results that have been proved already. Consequently, the tools that one has available for use in constructing proofs will vary, depending on how far the development of the system has gone. But the idea is that in natural deduction this should not be necessary, and *every* proof should rely just on the handful of rules first given as the rules of the system, even in practice. In other words, the initial rules should themselves be natural, and it should be natural to use them, and *only* them, in all one's deductions. That is primarily what we mean by 'natural' deduction.

As we shall see, there is some conflict between these requirements of naturalness and the requirements of elegance noted earlier.

EXERCISE

6.1.1. Using just the rules ($\wedge$I) and ($\wedge$E), and setting out proofs as in Chapter 5, give proofs of

(*a*) $P \wedge Q \dashv\vdash Q \wedge P$.
(*b*) $P \wedge (Q \wedge R) \dashv\vdash (P \wedge Q) \wedge R$.
(*c*) $P \dashv\vdash P \wedge P$.

7

Sequent Calculi

7.1. **The Idea** 273
7.2. **Natural Deduction as a Sequent Calculus** 277
7.3. **Semantic Tableaux as a Sequent Calculus** 283
7.4. **Gentzen Sequents; Semantic Tableaux Again** 291
7.5. **Comparison of Systems** 299
7.6. **Reasoning with Gentzen Sequents** 307

7.1. The Idea

Let us think, in a general way, about what happens in a natural deduction proof. As a whole the proof is an array of formulae, which we say establishes some sequent (namely the sequent which has on its left all the formulae which are undischarged assumptions in the proof, and on its right the single formula proved at the bottom of the proof). Moreover, the rules of inference too are rules about sequents. A proof always starts with an assumption, say φ, and if we add nothing more, then this itself counts as the proof of a sequent, namely

$$\varphi \vdash \varphi.$$

So the rule which allows us to get started is a rule which tells us directly that all sequents of this kind are correct. The other rules are all conditional, for they tell us that if certain sequents are correct, then so also is a further sequent, for example Modus Ponens in the form

If $\Gamma \vdash \varphi$ and $\Delta \vdash \varphi \rightarrow \psi$ then $\Gamma, \Delta \vdash \psi$.

So what happens in a proof is that we begin with certain sequents known to be correct, and we deduce that certain other sequents must therefore be correct. The proof proceeds by establishing one sequent after another, for at every step there is some sequent which is there established.

The idea of a sequent calculus is that it keeps an explicit record of just what sequent is established at each point of a proof. It does this by means of a new kind of proof in which every line is *itself* the sequent proved at that point in the proof. So a proof in a sequent calculus is not a linear sequence or other array *of formulae*, but a matching array *of whole sequents*. That is the basic idea.

Now we are familiar with sequents which have the (syntactic) turnstile $\vdash$ as their main verb; these are interpreted as claiming the existence of a proof, in whatever system of proof is currently being considered. We are also familiar with sequents which have the semantic turnstile $\vDash$ as their main verb; these make a claim about interpretations, namely that there is no interpretation which makes what is on the left true and what is on the right false. But neither of these signs has been allowed to occur *in* a proof. By convention, when we do have whole sequents occurring in a proof they are written not with $\vDash$ as their main verb, nor with $\vdash$, but instead with the new sign $\Rightarrow$. But the *intended* interpretation is that in which $\Rightarrow$ is taken to mean the same as the familiar turnstile $\vDash$. Consequently, $\Rightarrow$ has the same syntax as $\vDash$; it cannot occur within a formula but only between formulae, i.e. with some (or none) to the left and at the moment with just one to the right. There are, however, a couple of small changes that we must now make in our account of what a sequent is, and it is convenient to associate them with the change of notation.

The changes are required because it is a generally accepted condition on what can be counted as a proof that there must always be a mechanical decision procedure which can be applied to tell us whether or not an array of symbols is a proof.[1] An evident corollary of this is that a proof must be finite. Now a finite array of formulae is as a whole a finite structure, to which a decision procedure can be applied, because each formula is itself finite. But in a sequent calculus a proof is an array of sequents, not of formulae, so we must now insist that the sequents to be considered are themselves finite. That is the first change. Hitherto a sequent has been regarded as having a *set* of formulae on the left, and there has been no bar on infinite sets, but for the purposes of the present chapter they are debarred. As we saw in Chapter 4, nothing is actually lost thereby. For the compactness theorem (Section 4.8)

[1] In more *advanced* logic this condition is sometimes relaxed; but in elementary logic it is universally obeyed.

tells us that if we do have an infinite set of formulae which entails some formula φ, that is always because it has a finite subset which entails φ. The second change is a further elaboration of the first. While we regard a sequent as having a set of formulae on its left, we must accept that there are all kinds of ways of specifying such sets. For example, one could specify the set as: 'all formulae which will ever be written down by any person born on a Thursday'. No doubt that is a finite set, so a sequent given in this way would pass our first condition. But if proofs are to be certifiable as such by a mechanical decision procedure, then they clearly cannot be allowed to contain sequents given in this kind of way. We must instead require that what occurs to the left of ⇒ is to be a finite list, consisting of zero or more formulae, written out in full and separated by commas. In this chapter, the Greek letters 'Γ,Δ,...' will be used to represent such lists.

We may continue, if we like, to think of these lists of formulae as tacitly surrounded by curly brackets {. . .}, so that their role is still to specify a set. But now that we have come so far why should we not go one step further, and say that what is to the left of the ⇒ is not a set at all, but simply a finite sequence of zero or more formulae separated by commas? The answer is that we can perfectly well take this further step, though it does bring with it the need for an explicit statement of two further rules of inference. When a set is specified by listing its members, then the order in which the members are listed makes no difference, and any repetitions in the list may automatically be discounted. This is because sets are the same iff their members are the same, and different listings may yet list the same members. But if we are no longer thinking in terms of sets, and are working with lists directly, then we cannot continue with the attitude that it simply goes without saying that order and repetition are irrelevant. This is not a problem. It just means that we have to say it, instead of letting it go without saying. So we shall need two new rules of inference, the rule of Interchange (INT), which allows us to change the order of the formulae in the list, and the rule of Contraction, (CONTR), which allows us to delete a repetition.

It is customary to present a sequent calculus as a system in which proofs have the structure of trees, in the same way as we did first present natural deduction (in Section 6.2). At the topmost position on each branch there will therefore be a sequent which, according to the rules, can be asserted outright. This will therefore be an instance of the rule of assumptions. Every other position in the proof will be occupied by a sequent which is deduced from other sequents, the sequents that it is deduced from being written immediately above it, and separated from it by a horizontal line. We may therefore set out our basic rules of inference in the same way. Here, then, are the so-called 'structural' rules of inference, i.e. rules which do not concern any particular truth-functors or quantifiers.

$$\text{(ASS)}\ \frac{}{\varphi \Rightarrow \varphi}$$

$$\text{(THIN)}\ \frac{\Gamma \Rightarrow \varphi}{\Gamma,\psi \Rightarrow \varphi}$$

$$\text{(CUT)}\ \frac{\Gamma \Rightarrow \varphi \qquad\qquad \varphi,\Delta \Rightarrow \psi}{\Gamma,\Delta \Rightarrow \psi}$$

$$\text{(INT)}\ \frac{\Gamma,\varphi,\psi,\Delta \Rightarrow \chi}{\Gamma,\psi,\varphi,\Delta \Rightarrow \chi}$$

$$\text{(CONTR)}\ \frac{\Gamma,\varphi,\varphi \Rightarrow \chi}{\Gamma,\varphi \Rightarrow \chi}.$$

Whether one lists INT and CONTR explicitly as rules, or whether one lets them go without saying, is very much a matter of taste. (My own taste is to say that they should be listed as rules that are needed in theory, but then to let them go without saying in practice, since it is so very tedious to put in a separate step each time that one of them should, in theory, be invoked.) In any case, every sequent calculus will certainly conform to INT and CONTR, whether or not they are officially listed. But of the other rules one can only say that you would *expect* a sequent calculus to contain each of them (either as a basic rule or as derived from other basic rules). They all are basic rules in the system to be considered in the next section. But, as we shall see later on, there are sequent calculi in which they are either modified or lacking altogether.

EXERCISES

7.1.1.(*a*) Starting with an instance of ASS, and using suitable steps of THIN and INT, given in full, establish the sequent

$$P,Q,\neg P,\neg Q,R \Rightarrow \neg Q.$$

(*b*) By suitable steps of INT and CONTR, given in full, establish the following rule of inference

$$\frac{\varphi,\psi,\varphi,\chi,\psi \Rightarrow \neg\varphi}{\chi,\psi,\varphi \Rightarrow \neg\varphi}.$$

(*c*) Generalize your arguments in parts (*a*) and (*b*) to show that, however the formulae to the left of ⇒ may be listed initially, (1) the order of the list may be rearranged in any desired way, and (2) repetitions may be introduced or eliminated in any desired way, still leaving a sequent that is interdeducible with the one initially given.

7.1.2. A rule such as (∧I) is sometimes formulated in this way.

$$\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}$$

and sometimes in this way

$$\frac{\Gamma \Rightarrow \varphi \quad \Delta \Rightarrow \psi}{\Gamma,\Delta \Rightarrow \varphi \wedge \psi}$$

Show that, if Γ and Δ are both finite, then each of these formulations may be deduced from the other. [For the argument in one direction you will need THIN and INT; for the other direction you will need INT and CONTR.]

7.2. Natural Deduction as a Sequent Calculus

It is very simple to rewrite the natural deduction rules given in the last chapter as rules for a sequent calculus. We may adopt all the structural rules just noted, and then we may reproduce the rules for truth-functors and quantifiers, as given on pp. 252–4, in this form:[2]

$$(\wedge I)\ \frac{\Gamma \Rightarrow \varphi \quad \Delta \Rightarrow \psi}{\Gamma,\Delta \Rightarrow \varphi \wedge \phi} \qquad (\wedge E)\ \frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \varphi}, \quad \frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \psi}$$

$$(\vee I)\ \frac{\Gamma \Rightarrow \phi}{\Gamma \Rightarrow \varphi \vee \psi}, \quad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi} \qquad (\vee E)\ \frac{\Gamma \Rightarrow \varphi \vee \psi \quad \Delta,\varphi \Rightarrow \chi \quad \Theta,\psi \Rightarrow \chi}{\Gamma,\Delta,\Theta \Rightarrow \chi}$$

$$(\rightarrow I)\ \frac{\Gamma,\varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \rightarrow \psi} \qquad (\rightarrow E)\ \frac{\Gamma \Rightarrow \varphi \quad \Delta \Rightarrow \varphi \rightarrow \psi}{\Gamma,\Delta \Rightarrow \psi}$$

[2] Observe that the versions of (∨E) and (∃E) given here are more complex than those cited previously. The added complexity gives no further power (Exercise 7.2.1), but is adopted here because it better matches the way that (∨E) and (∃E) are actually used in natural deductions.

$$(\text{TND})\frac{\Gamma,\varphi \Rightarrow \psi \quad \Delta,\neg\varphi \Rightarrow \psi}{\Gamma,\Delta \Rightarrow \psi} \qquad (\text{EFQ})\frac{\Gamma \Rightarrow \varphi \quad \Delta \Rightarrow \neg\varphi}{\Gamma,\Delta \Rightarrow \psi}$$

$$(\forall\text{I})\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \forall\xi\varphi(\xi/\alpha)} \qquad (\forall\text{E})\frac{\Gamma \Rightarrow \forall\xi\varphi}{\Gamma \Rightarrow \varphi(\alpha/\xi)}$$

provided α is not in Γ

$$(\exists\text{I})\frac{\Gamma \Rightarrow \varphi(\alpha/\xi)}{\Gamma \Rightarrow \exists\xi\varphi} \qquad (\exists\text{E})\frac{\Gamma \Rightarrow \exists\xi\varphi(\xi/\alpha) \quad \Delta,\varphi \Rightarrow \psi}{\Gamma,\Delta \Rightarrow \psi}$$

provided α is not in Δ or ψ

Given the familiar rules formulated in this new way, and given also the basic idea that a proof in a sequent calculus records, at each stage, the whole sequent that has been proved at that stage, it is really very easy to see how a proof, originally written as a proof in natural deduction, may now be rewritten as a proof in this sequent calculus. I give just one example for a detailed analysis. Turn back to the proof given on p. 258 of the sequent

$$\forall x\exists y(Fx\wedge Gy) \vdash \exists y\forall x(Fx\wedge Gy).$$

This proof is rewritten as a sequent calculus proof below. You will observe that, to save clutter, I have omitted the small signs to the left of each horizontal line saying which rule of inference is being applied at that line. As an exercise, restore those signs. It will be seen that the structure of this proof is exactly the same as the structure of the proof given on p. 258, on which it is modelled. Indeed the two proofs correspond perfectly, step by step,[3] and this is not just an accident which happens to hold for this particular example but not for others. It should be perfectly clear that the point holds quite generally. Since it really is very simple to rewrite a natural deduction proof as a proof in the corresponding sequent calculus, one could at this point pass on without more ado to the next topic. But perhaps it will be useful if I make two further observations at this point.

The first is that it is evidently very tedious to write out a proof in our sequent calculus, and especially if the proof is to be given in a tree structure. But we have already seen that tree proofs may be collapsed into linear proofs, and that much ink is saved thereby, so can we not apply the same idea to these new sequent calculus proofs too? The answer is that we certainly can, and that this does indeed economize on ink and paper. But this answer

[3] The correspondence would not be perfect if steps of interchange and contraction had been put in explicitly. As an exercise, put them in.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\forall x\exists y(Fx\wedge Gy) \Rightarrow \forall x\exists y(Fx\wedge Gy)}}{\forall x\exists y(Fx\wedge Gy) \Rightarrow \exists y(Fa\wedge Gy)} \qquad \dfrac{\overline{Fa\wedge Gb \Rightarrow Fa\wedge Gb}}{Fa\wedge Gb \Rightarrow Fa}}{\forall x\exists y(Fx\wedge Gy) \Rightarrow Fa} \qquad \dfrac{\overline{Fc\wedge Gb \Rightarrow Fc\wedge Gb}}{Fc\wedge Gb \Rightarrow Gb}}{\forall x\exists y(Fx\wedge Gy), Fc\wedge Gb \Rightarrow Fa\wedge Gb}}{\forall x\exists y(Fx\wedge Gy), Fc\wedge Gb \Rightarrow \forall x(Fx\wedge Gb)}}{\forall x\exists y(Fx\wedge Gy), Fc\wedge Gb \Rightarrow \exists y\forall x(Fx\wedge Gy)} \qquad \dfrac{\overline{\forall x\exists y(Fx\wedge Gy) \Rightarrow \forall x\exists y(Fx\wedge Gy)}}{\forall x\exists y(Fx\wedge Gy) \Rightarrow \exists y(Fc\wedge Gy)}}{\forall x\exists y(Fx\wedge Gy) \Rightarrow \exists y\forall x(Fx\wedge Gy)}$$

can be improved to one which is perhaps more interesting, namely that our existing method of writing natural deduction proofs in a linear form is *already* a way of writing the corresponding sequent calculus proofs in linear form. I illustrate the point with the same sequent as example. A linear proof, drawn up according to the method of Section 6.4, would look like this:

$\forall x\exists y(Fx\wedge Gy) \vdash \exists y\forall x(Fx\wedge Gy)$

1	(1)	$\forall x\exists y(Fx\wedge Gy)$	ASS
1	(2)	$\exists y(Fa\wedge Gy)$	1,∀E
3	(3)	$Fa\wedge Gb$	ASS
3	(4)	Fa	3,∧E
1	(5)	Fa	2,3–4,∃E
1	(6)	$\exists y(Fc\wedge Gy)$	1,∀E
7	(7)	$Fc\wedge Gb$	ASS
7	(8)	Gb	7,∧E
1,7	(9)	$Fa\wedge Gb$	5,8,∧I
1,7	(10)	$\forall x(Fx\wedge Gb)$	9,∀I
1,7	(11)	$\exists y\forall x(Fx\wedge Gy)$	10,∃I
1	(12)	$\exists y\forall x(Fx\wedge Gy)$	6,7–11,∃E

Because each line in this proof contains, on the left, an explicit mention of the assumptions that the formula in that line rests on, we can easily see each line as being itself a sequent, namely the sequent which has the listed assumptions to its left and the formula displayed to its right. When we look at the proof in this way, we see that it is already a proof in a sequent calculus, and the justification for each line is unaffected. In fact it is just short for this explicit sequent calculus version:

(1)	$\forall x\exists y(Fx\wedge Gy) \Rightarrow \forall x\exists y(Fx\wedge Gy)$	ASS
(2)	$\forall x\exists y(Fx\wedge Gy) \Rightarrow \exists y(Fa\wedge Gy)$	1,∀E
(3)	$Fa\wedge Gb \Rightarrow Fa\wedge Gb$	ASS
(4)	$Fa\wedge Gb \Rightarrow Fa$	3,∧E
(5)	$\forall x\exists y(Fx\wedge Gy) \Rightarrow Fa$	2,3–4,∃E
(6)	$\forall x\exists y(Fx\wedge Gy) \Rightarrow \exists y(Fc\wedge Gy)$	1,∀E
(7)	$Fc\wedge Gb \Rightarrow Fc\wedge Gb$	ASS
(8)	$Fc\wedge Gb \Rightarrow Gb$	7,∧E
(9)	$\forall x\exists y(Fx\wedge Gy), Fc\wedge Gb \Rightarrow Fa\wedge Gb$	5,8,∧I
(10)	$\forall x\exists y(Fx\wedge Gy), Fc\wedge Gb \Rightarrow \forall x(Fx\wedge Gb)$	9,∀I
(11)	$\forall x\exists y(Fx\wedge Gy), Fc\wedge Gb \Rightarrow \exists y\forall x(Fx\wedge Gy)$	10,∃I
(12)	$\forall x\exists y(Fx\wedge Gy) \Rightarrow \exists y\forall x(Fx\wedge Gy)$	6,7–11,∃E

Given a natural deduction proof in the linear form, then, it is even more simple to rewrite it as a linear proof in the corresponding sequent calculus. In fact it is so simple that we may perfectly well count the original proof *as* a proof in the sequent calculus, but one that uses a convenient technique of abbreviation. For practical purposes, this is by far the best approach. But for most of the present chapter we shall not be too much concerned over what is convenient in practice, for one does not usually introduce a sequent calculus for that purpose. Rather, the interest is theoretical. As we shall see, a sequent calculus is a useful tool for comparing two systems that at first look utterly different. And for this purpose it is probably more helpful to stick to the original way of writing a proof, namely as a tree structure.

A second point, worth adding here, concerns sequents with no formula on the right. If we start from the perspective of natural deduction, we might expect such sequents to be defined in terms of the more familiar sequents with just one formula on the right. The simplest method of doing this is to suppose that the language already contains the symbol $\bot$, with its own rule of inference

$$(\bot)\frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow \varphi}.$$

Then clearly we can define

$$\Gamma \Rightarrow \quad \text{as short for} \quad \Gamma \Rightarrow \bot.$$

Alternatively, if $\bot$ is not available, but we do have (say) $\wedge$ and $\neg$, then we could instead define

$$\Gamma \Rightarrow \quad \text{as short for} \quad \Gamma \Rightarrow P \wedge \neg P.$$

It is artificial to pick on some particular contradictory formula to play this role, for no good ground could be given for choosing one rather than another, but in practice it works perfectly well. As a further alternative, we may, of course, accept sequents with no formula on the right as part of our primitive vocabulary, extending the usual structural rules to cover such sequents. Thus Thinning, Cutting, Interchange, and Contraction are to apply as before both when there is a formula on the right and when there is not, and there is also to be a new rule of Thinning *on the right*, which takes this form:

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \varphi}.$$

EXERCISES

7.2.1.(*a*) Assuming the standard structural rules, show that the rules ($\vee$E) and ($\exists$E) given above are interdeducible with these more familiar versions, which are rules for introducing on the left:

$$(\vee E')\ \frac{\Gamma,\varphi \Rightarrow \chi \quad \Delta,\psi \Rightarrow \chi}{\Gamma,\Delta,\varphi\vee\psi \Rightarrow \chi} \qquad (\exists E')\ \frac{\Gamma,\varphi \Rightarrow \psi}{\Gamma,\exists\xi\varphi(\xi/\alpha) \Rightarrow \psi}$$

provided α is not in Γ or in ψ

(*b*) Assuming the standard structural rules again, show that the rule ($\forall$E) given above is interdeducible with this rule for introducing on the left:

$$(\forall E')\ \frac{\Gamma,\varphi(\alpha/\xi) \Rightarrow \psi}{\Gamma,\forall\xi\varphi \Rightarrow \psi}.$$

(*c*) Consider once more the sequent

$$\forall x\exists y(Fx\wedge Gy) \Rightarrow \exists y\forall x(Fx\wedge Gy).$$

Construct a proof of this sequent using ($\exists$E′) and ($\forall$E′) in place of ($\exists$E) and ($\forall$E), and *not* using CUT.

7.2.2. Let us write a double horizontal line to signify that the sequent below the line follows from the ones above *and* conversely that the ones above each follow from the one below. Then what were called in Section 2.5 the 'basic principles' for $\wedge,\vee,\rightarrow,\neg$ may be formulated thus:

$$\frac{\Gamma \Rightarrow \varphi \quad \text{AND} \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi\wedge\psi} \qquad \frac{\Gamma,\varphi \Rightarrow \chi \quad \text{AND} \quad \Gamma,\psi \Rightarrow \chi}{\Gamma,\varphi\vee\psi \Rightarrow \chi}$$

$$\frac{\Gamma,\varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi\rightarrow\psi} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma,\neg\varphi \Rightarrow}$$

Assuming all the standard structural rules for a sequent calculus, show that these rules are interdeducible with the ones given in this section.

7.2.3. Consider a sequent calculus which has all the standard structural rules and in addition just this one pair of rules:

$$\frac{\Gamma \Rightarrow \varphi \quad \text{AND} \quad \Gamma \Rightarrow \psi}{\Gamma,\varphi\uparrow\psi \Rightarrow}.$$

(*a*) Give an interpretation of $\uparrow$ under which this rule is sound.
(*b*) Define $\neg, \wedge, \vee$ in terms of $\uparrow$, and show how to prove, in this calculus, the rules for $\neg, \wedge, \vee$ given in this section.