8.1. Identity

We shall use '$a=b$' as short for 'a is the same thing as b'. The sign '=' thus expresses a particular two-place predicate, and since we generally write a predicate-symbol in front of the name-letters that fill its gaps, you might have expected the same here. Very occasionally this can be convenient (Exercise 8.1.2), but it is confusing to have the same sign '=' appearing in these two roles. So let us say that officially the letter 'I' is the identity predicate, and it is to have just the same grammar as the familiar two-place predicate-letters. For example, 'Iab' is a formula. But almost always we shall 'abbreviate' this formula to '$a=b$'. Similarly, we shall abbreviate the formula '$\neg Iab$' to '$a \neq b$'.

It is easy to see how to incorporate the new symbol into our formal languages. First, the formation rules are extended, so that they include a clause stating that, if τ_1 and τ_2 are any terms (i.e. names or variables) then $I\tau_1\tau_2$ (or $\tau_1 = \tau_2$) is a formula. Second, the intended meaning of this symbol is reflected in a suitable rule for interpretations of the language. An interpretation I is said to be a *normal* interpretation iff it satisfies the condition that, for any name-letters α and β,

$$|\alpha = \beta|_I = \mathrm{T} \quad \text{iff} \quad |\alpha|_I = |\beta|_I.$$

Alternatively, if our interpretations are specified by a recursion on satisfaction rather than truth, then the relevant condition is

$$\sigma \text{ sats } \tau_1 = \tau_2 \quad \text{iff} \quad \sigma(\tau_1) = \sigma(\tau_2).$$

Given this intended interpretation, it is clear that we have as correct theses for identity

$$\models \alpha{=}\alpha$$
$$\alpha{=}\beta \models \varphi(\alpha/\xi) \leftrightarrow \varphi(\beta/\xi).$$

(Recall that the consequent of the second thesis means: if you start with a formula containing occurrences of α, and substitute β for some, but not necessarily all, of those occurrences, then the two formulae have the same truth-value.) These two together are usually taken as the basic principles for identity. With scant regard for history, the second of them is often called Leibniz's law, but the first has no special name (except that once upon a time it was called 'the' law of identity).

There is much more that could be said about ordering relations—a little of it will emerge from Exercise 8.1.2—but I do not pursue this topic further. Instead I mention another important way in which identity is used in the classification of relations, namely in the definition of what is called a *one–one* relation. This is the amalgamation of two simpler conditions. We say that a relation R is *one–many* iff

$$\forall xyz(Rxz \wedge Ryz \rightarrow x=y).$$

and it is *many–one* iff

$$\forall xyz(Rzx \wedge Rzy \rightarrow x=y).$$

(You will see that 'one–many' means, in effect 'for anything on the right there is at most one on the left', whereas 'many–one' means 'for anything on the left there is at most one on the right'.) A one–one relation is one that satisfies both of these conditions, i.e. it is both one–many and many–one. A neat way of amalgamating the two conditions is

$$\forall xyzw(Rxz \wedge Ryw \rightarrow (x=y \leftrightarrow z=w)).$$

These ideas will recur in what follows, so I do not develop them any further now. Let us turn instead to our other topic involving identity, namely the 'numerical quantifiers'.

To say that there is at least one thing x such that Fx we need only use an existential quantifier.

$$\exists xFx.$$

To say that there are least two such things we need identity as well, as in

$$\exists x(Fx \wedge \exists y(Fy \wedge y \neq x)).$$

Similarly, to say that there are at least three we need a formula such as

$$\exists x(Fx \wedge \exists y(Fy \wedge y \neq x \wedge \exists z(Fz \wedge z \neq y \wedge z \neq x))).$$

It is clear that there is a pattern in these formulae. Using '$\exists_n x$' to mean 'there are at least n things x such that', and using 'n'' for 'the number after n' we can sum up the pattern in this way:

$$\exists_1 x(Fx) \leftrightarrow \exists xFx$$
$$\exists_{n'} x(Fx) \leftrightarrow \exists x(Fx \wedge \exists_n y(Fy \wedge y \neq x)).$$

One can use this pattern to define any specific numeral in place of 'n'. Interestingly, we find the *same* pattern when we look into 'exactly n' rather than 'at least n'. If we represent 'there are exactly n things x such that' by the simple 'nx', we have

$$0x(Fx) \leftrightarrow \neg\exists xFx$$
$$n'x(Fx) \leftrightarrow \exists x(Fx \wedge ny(Fy \wedge y \neq x)).$$

Using these definitions, one can represent in 'purely logical' vocabulary such apparently 'arithmetical' theses as

$$2x(Fx \wedge Gx) \wedge 3x(Fx \wedge \neg Gx) \rightarrow 5x(Fx).$$

One can prove such theses too, by 'purely logical' means, assuming that our rules for identity are counted as a part of 'pure logic'. But we shall leave it to the philosophers to dispute about the relationship between this thesis and the genuinely arithmetical thesis

$$2 + 3 = 5.$$

8.1.3.(*a*) What would be wrong with the following scheme for defining the numerical quantifier 'there are at least *n*'?

$$\exists_1 x(Fx) \leftrightarrow \exists x Fx$$
$$\exists_{n'} x(Fx) \leftrightarrow \exists_n x(Fx \wedge \exists y(Fy \wedge y \neq x)).$$

(*b*) Suppose that new numerical quantifiers $\forall_n$ are defined by the scheme

$$\forall_0 x(Fx) \leftrightarrow \forall x \neg Fx$$
$$\forall_{n'} x(Fx) \leftrightarrow \forall x(\neg Fx \vee \forall_n y(Fy \wedge y \neq x)).$$

What is the right interpretation of these quantifiers?