8.3. Descriptions

A functional expression such as 'the father of *a*' is a special case of what is called a *definite description*. This is a singular noun-phrase, beginning with the definite article 'the', which one might naturally think of as purporting to refer to just one thing. (In a sentence such as 'The seventh child is most likely to have second sight', the phrase 'the seventh child' is a definite description

[1] Further hint, to be consulted if really needed:

Take the domain to consist of both the natural numbers and the signed integers, and assume that no natural number is itself a signed integer.

if, in the context, there is some particular child that is referred to, but not if the remark is intended as a generalization over all seventh children.) We can form a definite description out of any one-place predicate 'Fx' by adding a suitable prefix at the front, as in 'the thing x such that Fx'. Functional expressions are expressions of this kind, but in their case the object is always described by means of a one–many relation that it bears to some other object (or pair of objects, etc.) For example, 'the father of a' is easily seen as short for 'the thing x such that x fathered a', and similarly '$a + b$' is easily seen as short for 'the number x which results upon adding a and b'. Definite descriptions do quite often have this structure, but they do not have to. For example, 'the only man here with a blue beard' is a perfectly good definite description, but it is not naturally seen as involving any one–many relation.

It is quite natural to suppose that one uses a definite description only when one believes that, in the context, it describes one and only one thing. But (*a*) a little reflection shows that there are clear exceptions to this generalization. For example, one who says 'There is no such thing as the greatest prime number' is using the definite description 'the greatest prime number', but not because he believes that there is some one thing that it describes. Besides (*b*) even if the speaker does *believe* that his description singles out some one thing, still he may be mistaken. For example, I may say to someone, in all sincerity, 'I saw your dog in the park yesterday, chasing squirrels'. The expression 'your dog' is clearly a definite description (short for 'the dog that belongs to you'), and no doubt I would not have said what I did unless I believed that the person in question owned a dog. But I may have got it all wrong, and perhaps that person has never owned a dog. In that case I have made a definite claim, but a false claim, for I could not have seen your dog if in fact there is no such thing.

Even in ordinary speech, then, we do in fact use definite descriptions which fail to refer. And if we are going to admit descriptions into our logic, then we certainly cannot overlook this possibility. For you can form a definite description out of any one-place predicate whatever, by prefixing to it the words 'the x such that', but it would be idiotic to suppose that every one-place predicate is satisfied by exactly one object. So although a definite description looks like a name (a complex name), and in many ways behaves like a name, still it cannot *be* a name if names must always refer to objects. We therefore need some other way of handling these expressions, and it was Bertrand Russell (1905) who first provided one.

Russell introduces the notation

$$(\iota x{:}Fx)$$

to represent 'the x such that Fx'. (The symbol ι is a Greek iota, upside-down.) In his symbolism this expression can take the place of a name, as in the formula

$$G(\iota x{:}Fx).$$

But although the expression behaves like a name in this way, still Russell's theory is that it is not really a name. For the expression is introduced by a definition which stipulates that this whole formula is short for

$$\exists y(\forall x(Fx \leftrightarrow x{=}y) \wedge Gy).$$

That is, the formula makes this complex claim: 'There is one and only one thing such that F (it), and in addition G (that thing)'. Consequently, when the definite description is not uniquely satisfied, the whole formula is false.

That is only a rough outline of Russell's theory, and we soon see that more is needed. For consider the formula

$$\neg G(\iota x{:}Fx).$$

Here we must choose whether to apply the proposed analysis just to the part $G(\iota x{:}Fx)$, or to the whole formula. In the first case the $\neg$ remains at the front, undisturbed by the analysis, and we obtain

$$\neg\exists y(\forall x(Fx \leftrightarrow x{=}y) \wedge Gy).$$

In the second case the $\neg$ is itself taken into the analysis, and we get

$$\exists y(\forall x(Fx \leftrightarrow x{=}y) \wedge \neg Gy).$$

In the first case we say, for obvious reasons, that the $\neg$ has major scope (or wide scope), and the definite description has minor scope (or narrow scope); in the second case we say that the definite description has major scope, and the $\neg$ has minor scope. Evidently it can make a difference whether we assign the scope in one way or the other. So this at once reveals an important way in which definite descriptions differ from names on this theory, for in orthodox logic names are not regarded as having scope, whereas on Russell's theory definite descriptions certainly do. We need, then, some way of representing these scopes in our formal notation.

Russell had his own way (which I shall come to in a moment), but I think that nowadays the preferred method is to say that it was a mistake in the first place to allow definite descriptions to take the place of names. After all, if they have scopes while names do not, this must lead to trouble. So the suggestion is that the main idea behind Russell's analysis is much better presented if we 'parse' definite descriptions not as names but as quantifiers, for we all know that quantifiers must have scopes. On this proposal, the definite article 'the' is to be treated as belonging to the same category as the acknowledged quantifier-expressions 'all' and 'some'. In English all three of these expressions occur in prefixes to noun-clauses which can take the place of names, but which in standard logical notation are pulled to the front of the open sentences which represent their scopes, precisely in order to reveal what the scope is. We can think of the analysis in this way. If we start with

All men are mortal,

then the scope of the quantifying phrase 'all men' may be explicitly represented in this way:

$(\forall \text{ men } x)(x \text{ is mortal})$.

And then the structure of this quantifying phrase itself may be revealed by rewriting it in this form:

$(\forall x{:}x \text{ is a man})(x \text{ is mortal})$,

where the colon ':' abbreviates 'such that'. We can repeat the same suggestions both for '∃', representing 'some', and for the new quantifier 'I' which I now introduce as representing 'the' when regarded as a quantifier. ('I' is, of course, a capital Greek iota, written both upside-down and back to front, as befits a quantifier.) So we now have sentences of the pattern

$(\forall x{:}Fx)(Gx)$.
$(\exists x{:}Fx)(Gx)$.
$(\mathrm{I}x{:}Fx)(Gx)$.

The prefixes are to be read, respectively, as

For all x such that Fx
For some x such that Fx
For the x such that Fx.

These prefixes are restricted quantifiers, and as a final step they in turn may be analysed in terms of the unrestricted quantifiers used in elementary logic. Upon this analysis, the three sentence-patterns we began with are transformed into

$\forall x(Fx \rightarrow Gx)$.
$\exists x(Fx \wedge Gx)$.
$\exists x(\forall y(Fy \leftrightarrow y{=}z) \wedge Gx)$.

If 'the' is regarded in this way as a quantifier, then, of course, it will make a difference whether we write $\neg$ before or after $(\mathrm{I}x{:}Fx)$, just as it makes a difference whether we write it before or after $(\forall x{:}Fx)$ and $(\exists x{:}Fx)$. But I remark here that in the case of I it makes a difference only when the definite description is not uniquely satisfied. Let us borrow Russell's notation once more, and abbreviate

$$\exists x \forall y(Fy \leftrightarrow y=z)$$

to

$$E!(\iota x{:}Fx)$$

('$E!$' is for 'exists'). Then it is perfectly simple to prove that

$$E!(\iota x{:}Fx) \models (\mathrm{I}x{:}Fx)(\neg Gx) \leftrightarrow \neg(\mathrm{I}x{:}Fx)(Gx).$$
$$E!(\iota x{:}Fx) \models (\mathrm{I}x{:}Fx)(P \wedge Gx) \leftrightarrow P \wedge (\mathrm{I}x{:}Fx)(Gx).$$
$$E!(\iota x{:}Fx) \models (\mathrm{I}x{:}Fx)(Gx \wedge Hx) \leftrightarrow (\mathrm{I}x{:}Gx)(Gx) \wedge (\mathrm{I}x{:}Fx)(Hx).$$
$$E!(\iota x{:}Fx) \models (\mathrm{I}x{:}Fx)\forall y(Rxy) \leftrightarrow \forall y(\mathrm{I}xFx)(Rxy).$$

In fact the last three of these sequents hold without the premiss $E!(\iota x{:}Fx)$, but in place of the first we then have

$$\neg E!(\iota x{:}Fx) \models \neg(\mathrm{I}x{:}Fx)(Gx)$$

and hence also

$$\neg E!(\iota x{:}Fx) \models \neg(\mathrm{I}x{:}Fx)(\neg Gx).$$

From a technical point of view, there is nothing wrong with the definition of I just introduced, but it is extremely tedious in practice. I illustrate with a couple of examples from arithmetic. Suppose that we have a couple of three-place predicates representing addition and multiplication thus:

Sxyz: adding *x* and *y* yields *z*.
Pxyz: multiplying *x* and *y* yields *z*.

From these we can, of course, form the definite description quantifiers

$$(\mathrm{I}z{:}Sxyz)$$
$$(\mathrm{I}z{:}Pxyz).$$

And we can use these to analyse arithmetical sentences containing '+' and '·'. For example, the simple statement

$$a + b = b + a$$

can be analysed in this way:

$$(\mathrm{I}x{:}Sabx)(\mathrm{I}y{:}Sbay)(x=y).$$

This is already a little unexpected, but—one might say—perhaps we could get used to it. Consider, then, something just a bit more complicated, such as

$$a \cdot (b+c) = a \cdot b + a \cdot c.$$

After some thought, one sees that the analysis must be this:

$$(\mathrm{I}x{:}Sbcx)(\mathrm{I}y{:}Paby)(\mathrm{I}z{:}Pacz)(\mathrm{I}u{:}Paxu)(\mathrm{I}v{:}Syzv)(u=v).$$

And with the present notation there appears to be no way of introducing any simplifications.

Contrast with this what the position would have been if we had retained the original symbol ι, conceived as an operator which produces from an open sentence, not a quantifier, but a (complex) name. We could then have written

$$a+b=b+a$$

in the simpler form

$$(\iota x{:}Sabx) = (\iota x{:}Sbax).$$

And we could indeed have returned to the original notation, which is simpler still, by saying: let us *abbreviate*

$$a+b \quad \text{for} \quad (\iota x{:}Sabx).$$

This abbreviation is available with $(\iota x{:}Sabx)$, since in this expression the variable x is bound by the prefix ιx, and no occurrences of x outside the expression are relevant to its interpretation. Hence an abbreviation may omit the variable x altogether. But we cannot do the same with the quantifier $(\mathrm{I}x{:}Sabx)$, since this quantifier is used to bind *further* occurrences of x. Consequently, if an abbreviation omits x from $(\mathrm{I}x{:}Sabx)$, then the rest of the formula must fall into confusion. Thus the quantifying notation, apparently needed in order to represent scopes explicitly, also prevents one from using the simple and traditional notation of ordinary mathematics.

Russell himself proposed an ingenious way out of the difficulty. Suppose that we begin with definite descriptions construed as variable-binding quantifiers. Thus, to continue with the same example, the description 'the x such that $Sabx$' occurs in contexts of the kind

$$(\mathrm{I}x{:}Sabx)(\text{—}x\text{—}x\text{—}).$$

Then, to obtain Russell's own notation, for each subsequent occurrence of x, bound by $(\mathrm{I}x{:}Sabx)$, we write instead the namelike expression $(\iota x{:}Sabx)$. At the same time we write ɩ in place of I in the original quantifier, but change its surrounding brackets to square brackets, in order to show which occurrences of this expression are quantifiers and which are not. Thus we reach

$$[\iota x{:}Sabx](\text{—}(\iota x{:}Sabx)\text{—}(\iota x{:}Sabx)\text{—}).$$

This is the notation Russell uses himself, and, as I have indicated, it is easily seen as just a variation on the natural way of representing descriptions as quantifiers. But in practice it has two great advantages. First, there is now no obstacle to abbreviating a definite description in a way that omits its variable x. Thus, in place of the formula above we may write simply

$$[a+b](\text{—}(a+b)\text{—}(a+b)\text{—}).$$

Second, we may introduce a convention by which the initial quantifying expression $[\iota x{:}Sabx]$ or $[a+b]$ may be omitted altogether. Russell's basic idea here is that such a quantifier may be omitted when its scope is the smallest possible, i.e. when its scope is an atomic formula, but it must be shown explicitly when it includes in its scope either truth-functors or other quantifiers. In practice, however, he *also* allows the omission of a description-quantifier when its scope contains other *description*-quantifiers. Thus he would omit *both* description-quantifiers from

$$[a+b][b+a]((a+b) = (b+a))$$

and he would omit *all five* from

$$[b+c][a\cdot b][a\cdot c][a\cdot(b+c)][(a\cdot b)+(a\cdot c)]((a\cdot(b+c)) = ((a\cdot b)+(a\cdot c))).$$

To avoid ambiguity, then, we need further conventions to tell us in what order the description-quantifiers are to be restored when more than one has been omitted, and this order must take account of the fact that a complex description may contain another description inside itself, as the second example illustrates. But I shall not delay to formulate such conventions. The basic idea is, I hope, clear enough.

Russell's own procedure, then, is a very ingenious way of getting the best of both worlds. In practice, definite descriptions are for the most part treated as names, since this is by far the most convenient notation, but in theory they are treated as quantifiers, since in theory they are assigned scopes, and quantifiers have scopes whereas names do not. Moreover, theory and practice fit quite nicely with one another, because in contexts where we need to make *serious* use of descriptions we always assure ourselves first that the descriptions are uniquely satisfied. And when a description is uniquely satisfied then it does behave like a name, since all ways of assigning its scope are equivalent to one another.[2] But the theory will still accommodate descriptions that are not uniquely satisfied, just because in theory they are quantifiers and not names.

Despite Russell's ingenious compromise, I think one still feels that there is a tension in this theory, just because descriptions are officially introduced in one way but then treated as much as possible in a different way. The need for this tension arises because, although descriptions do behave very much as names do, still we began by saying that they could not actually *be* names, since a description can fail to refer but a name cannot. But it is now time to look once more at that very basic assumption: why must we say that a name cannot lack a reference?

EXERCISES

8.3.1. Prove by an informal argument, with just enough detail to be convincing, that the following formulae are all equivalent. (Hence any of them might have been used to provide a Russellian analysis of 'the *F* is *G*'.)

(1) $\exists x(\forall y(Fy \leftrightarrow y{=}x) \wedge Gx)$
(2) $\exists x\forall y(Fy \leftrightarrow y{=}x) \wedge \forall x(Fx \rightarrow Gx)$
(3) $\forall x\forall y(Fx{\wedge}Fy \rightarrow x{=}y) \wedge \exists x(Fx \wedge Gx)$.

8.3.2.(*a*) Prove, by any means you like, the sequents cited on p. 345.
(*b*) Show in detail how these justify the claim that, where a definite description is uniquely satisfied, all ways of assigning its scope are equivalent.

8.3.3. Taking the domain to be the real numbers, use the definite description quantifier I, and the predicates *Sxyz* and *Pxyz* of p. 345, to give an analysis of

(1) $a+(b+c) = (a+b)+c$.
(2) $(a+b)^2 = a^2 + 2ab + b^2$.
(3) $\frac{a}{b}+\frac{c}{d} = \frac{a{\cdot}d+b{\cdot}c}{b{\cdot}d}$.

Is (3), in your analysis, a true statement?