5.5. A Completeness Proof

We shall now show that the three axioms (A1)–(A3), together with the rule MP, form a complete basis for truth-functional logic. As we noted long ago, $\rightarrow$ and $\neg$ are together expressively adequate (p. 46), i.e. can express every truth-function. Also, as we showed in the last chapter, the logic of truth-functions is compact (pp. 173–4), so that sequents with infinitely many formulae on the left introduce nothing new. Moreover, if we are confining attention to finite sequents, then we can also confine our attention further to those with *no* formula on the left, since any finite sequent can be exchanged for its conditionalization (pp. 202–3). To prove the completeness of our system, then, it is sufficient to show that, for any formula φ,

If $\models \varphi$ then $\vdash \varphi$.

That is what we shall now show.

The idea of the proof is this. If the formula φ is valid then it is a truth-table tautology, i.e. it comes out true in every row of the truth-table. We shall show that our deductive system can *mirror* the truth-table calculations.[4] Suppose that we have a formula φ which contains just the sentence-letters $P_1,...,P_n$, and no others. Then a row of the truth-table says that, for a given assignment of truth-values to the letters $P_1,...,P_n$, the formula φ takes a certain value. We can say the same thing by means of a sequent of the form

$$\pm P_1,...,\pm P_n \vdash \pm\varphi,$$

where $\pm P_i$ is either P_i or $\neg P_i$, depending on whether P_i takes the value T or the value F in that row, and similarly $\pm\varphi$ is either φ or $\neg\varphi$, depending on whether φ then takes the value T or F in that row. Let us say that this sequent is the sequent that *corresponds* to that row of the truth-table. Our completeness proof will show as a first step that for each row of any truth-table the corresponding sequent is provable in our deductive system. Then as a second step it will show that a tautology, which comes out true in every row of its truth-table, is provable on no assumptions at all. For convenience I list here the lemmas that will be needed about our system. For the first part of the proof they are

(1) $\varphi \vdash \neg\neg\varphi$.
(2) $\varphi \vdash \psi\rightarrow\varphi$.
(3) $\neg\varphi \vdash \varphi\rightarrow\psi$.
(4) $\varphi, \neg\psi \vdash \neg(\varphi\rightarrow\psi)$.

[4] This proof is due to Kalmár (1934–5).

Of these, (1) was proved as (T7) of the previous section; (2) is an immediate consequence of the deduction theorem; (3) results by applying the deduction theorem to (T1) of the previous section; and the proof of (4) may safely be left as an exercise. In the second part of the proof we shall need a version of TND, and the most convenient form is this:

(5) If $\Gamma,\varphi \vdash \psi$ and $\Gamma,\neg\varphi \vdash \psi$ then $\Gamma \vdash \psi$.

This is obtained by applying the deduction theorem to (T10) of the previous section. I remark here that since lemmas (1)–(5) are the *only* features of the deductive system that are needed for this completeness proof, it follows that any other system which contains (1)–(5)—including the system which consists *just* of (1)–(5) and nothing else (except the structural rules)—is equally complete, in the sense that it suffices to yield a proof, on no assumptions, of every tautology. (But we would need to add the rule MP if we are to ensure that from a proof of the conditionalization of a sequent, we can always construct a proof of the sequent itself.)

As the first stage of the proof we need to show this. Let the letters $P_1,...,P_n$ be the letters in a formula φ. Consider any assignment of truth-values to those letters. Let $\pm P_i$ be P_i or $\neg P_i$ according as P_i is assigned the value T or the value F in that assignment. Then we must establish

(*a*) If φ is true on this assignment, then $\pm P_1,...,\pm P_n \vdash \varphi$.
(*b*) If φ is false on this assignment, then $\pm P_1,...,\pm P_n \vdash \neg\varphi$.

The proof is by induction on the length of the formula φ. We have three cases to consider:

Case (1): φ is atomic, say P_i. Then what has to be shown is

(*a*) $P_i \vdash P_i$
(*b*) $\neg P_i \vdash \neg P_i$.

This is immediate.

Case (2): φ is $\neg\psi$. Then the letters in φ are the same as those in ψ. (*a*) Suppose φ is true on the assignment. Then ψ is false, and by inductive hypothesis we have

$\pm P_1,...,\pm P_n \vdash \neg\psi$
i.e. $\pm P_1,...,\pm P_n \vdash \varphi$

as required. (*b*) Suppose φ is false. Then ψ is true, and by inductive hypothesis

$\pm P_1,...,\pm P_n \vdash \psi$.

Hence, by lemma (1)

$\pm P_1,...,\pm P_n \vdash \neg\neg\psi$
i.e. $\pm P_1,...,\pm P_n \vdash \varphi$

as required.

Case (3): φ is $\psi\rightarrow\chi$. Then the letters in ψ and in χ are subsets of those in φ, say $P_i,...,P_j$ and $P_k,...,P_l$ respectively. (*a*) Suppose φ is true on the assignment. Then either ψ is false or χ is true, and by inductive hypothesis

$\pm P_i,...,\pm P_j \vdash \neg\psi$ or $\pm P_k,...,\pm P_l \vdash \chi$
$\therefore$ $\pm P_1,...\pm P_n \vdash \neg\psi$ or $\pm P_1,...,\pm P_n \vdash \chi$ (by THIN)
$\therefore$ $\pm P_1,...\pm P_n \vdash \psi\rightarrow\chi$ (by lemmas (2) and (3))
i.e. $\pm P_1,...\pm P_n \vdash \varphi$, as required.

(*b*) Suppose φ is false on the assignment. Then ψ is true and χ is false, and by inductive hypothesis

$\pm P_i,...,\pm P_j \vdash \psi$ and $\pm P_k,...,\pm P_l \vdash \neg\chi$
$\therefore$ $\pm P_1,...,\pm P_n \vdash \psi$ and $\pm P_1,...,\pm P_n \vdash \neg\chi$ (by THIN)
$\therefore$ $\pm P_1,...,\pm P_n \vdash \neg(\psi\rightarrow\chi)$ (by lemma (4))
i.e. $\pm P_1,...,\pm P_n \vdash \varphi$, as required.

This completes the induction, and therefore completes the first stage of our proof. So we now move on to the second stage.

Suppose that φ is a tautology. If there are n letters in φ, then there are 2^n rows in the truth-table for φ, and for each of them there is a corresponding sequent which is provable. We consider these sequents in the order of the corresponding rows of the truth-table, and take them in pairs. Each pair has the form

$\pm P_1,...,\pm P_{n-1}, P_n \vdash \varphi.$
$\pm P_1,...,\pm P_{n-1}, \neg P_n \vdash \varphi.$

Applying lemma (5) to each pair, we infer in each case

$\pm P_1,...,\pm P_{n-1} \vdash \varphi.$

This leaves us a set of 2^{n-1} provable sequents, covering every assignment of truth-values to the sentence-letters $P_1,...,P_{n-1}$, but no longer containing the letter P_n on the left. By taking these in pairs, and applying lemma (5) to each pair, as before, we obtain a set of 2^{n-2} provable sequents, covering every assignment of truth-values to the letters $P_1,...,P_{n-2}$, but no longer containing the letters P_n or P_{n-1} on the left. By continuing this manœuvre, as often as necessary, we eventually reach the provable sequent $\vdash \varphi$, with no sentence-letters on the left. This completes the argument.

EXERCISES

5.5.1. Let I be the intuitionist logic specified in Exercise 5.4.4, and let $\vdash_I$ mean provability in that logic.
(*a*) Establish the lemmas

$$\varphi \vdash_I \neg\neg\varphi.$$
$$\varphi \vdash_I \psi\rightarrow\varphi.$$
$$\neg\varphi \vdash_I \varphi\rightarrow\psi.$$
$$\varphi,\neg\psi \vdash_I \neg(\varphi\rightarrow\psi).$$

Deduce that the first stage of our completeness proof holds also for intuitionist logic.
(*b*) Establish the lemma

$$\text{If } \Gamma,\varphi \vdash_I \neg\neg\psi \text{ and } \Gamma,\neg\varphi \vdash_I \neg\neg\psi \text{ then } \Gamma \vdash_I \neg\neg\psi.$$

Deduce that the second stage of our completeness proof can be modified to yield this result:

$$\text{If } \models \varphi \text{ then } \vdash_I \neg\neg\varphi.$$

5.5.2.(*a*) Show that an axiomatic system S which contains EFQ is absolutely consistent iff it is negation-consistent, i.e.

$$\text{for all } \varphi, \vdash_S \varphi \quad \text{iff} \quad \text{for some } \varphi, \vdash_S \varphi \text{ and } \vdash_S \neg\varphi.$$

(*b*) Show that if we add to the axioms (A1)–(A3) any new axiom-schema of our language, not already provable from those axioms, then the result is an inconsistent system. (That is, the axioms (A1)–(A3) are 'complete in the sense of Post (1921)'.) [Since (A1)–(A3) are complete, any axiom-schema not provable from them must be non-tautologous. So it has to be shown that any non-tautologous schema has an inconsistent substitution-instance.]