5

Axiomatic Proofs

5.1. The Idea

The idea of using semantic tableaux to provide a proof procedure is a recent invention. (In effect it stems from work done by Gentzen (1934), but the main ideas were first clearly presented by Beth (1955). The tree format used in the last chapter is due to Jeffrey (1981).) Originally proofs in elementary logic were quite differently conceived.

One of the great achievements of Greek mathematics was the introduction of 'the axiomatic method', most famously in Euclid's *Elements*, but by no means confined to that work. The method results quite naturally from reflection upon the idea of a proof. For in an ordinary proof one shows that some proposition is true by showing that it follows from premisses that are already accepted as true, and this will lead the theoretician to ask whether those premisses could be proved in their turn. Pressing this question, one is led to the idea of the 'basic premisses' of the subject, from which all other propositions must be proved, but which must themselves be accepted without proof. These, then, are the axioms of the subject. It was recognized by Aristotle that there must be such axioms for any subject that could be 'scientifically' pursued; and it was Euclid's achievement to have found a set of axioms from which almost all the mathematics then known could be deduced. In fact the axioms need some supplementation if the deduction is to confirm to modern standards of rigour, but that is of small importance. The Greeks had, apparently, supplied mathematics with a clear 'foundation'.

Over the succeeding centuries mathematics grew and developed in many ways, but it was not until the nineteenth century that interest turned once more to the question of 'foundations'. By then it was quite clear that Euclid's work would no longer suffice, and this led to a renewed search for the basic premisses of the subject. At the same time some mathematicians became interested in the principles of logic, which governed the deductions from these premisses, and an interest in both topics at once led Frege to the 'logicist' theory of the foundations of mathematics. This theory is that mathematics has no special axioms of its own, but follows just from the principles of logic themselves, when augmented by suitable definitions. To argue in detail for this theory Frege had first to supply an adequate account of the principles of logic, which he did in his *Begriffsschrift* of 1879. This epoch-making work was the first presentation of what we now think of as modern logic, and in it Frege supplied a set of axioms, i.e. basic premisses, for logic itself. No doubt he was led to present the foundations of logic in this way at least partly because it was a well-known way of presenting the foundations of other disciplines, especially parts of mathematics. But nowadays it does not strike us as at all natural for logic.

Logic has always been regarded as concerned with correct inference, and so it is natural to expect that it will take as its basic notion the relation of entailment between premisses and conclusion. In fact the proof technique of the last chapter did not conform to that expectation entirely. For the original version aims to show, quite generally, that certain combinations of truth-values are impossible, and while this includes entailment as a special case, it is not directly focused upon it. And the revised version, which one uses in practice, is naturally seen just as a technique for proving inconsistency. Equally a proof technique that is founded on axioms does not conform, since an axiom is basically a claim that something is true. More particularly, an axiom of logic claims that something is logically true, and hence a necessary truth. So when we employ 'formal' (i.e. schematic) languages in our logic, an axiom will claim that some formula is such that all its instances are logically true, i.e. that the formula comes out true under all (possible) interpretations of its non-logical symbols, which is to say that it is a valid formula. So here the basic use of the turnstile has one formula on the right and none on the left. Of course, facts about entailment will follow from this, for, as we know,

$$\varphi_1,\ldots,\varphi_n \vDash \psi \quad \text{iff} \quad \vDash (\varphi_1 \rightarrow \ldots(\varphi_n \rightarrow \psi)..)$$

But we do not work with entailment from the beginning. Our axioms are single formulae, not sequents of several formulae.

We shall lay down infinitely many axioms, which we do by using axiom-schemas. For example, our first axiom-schema will be

$$\varphi \rightarrow (\psi \rightarrow \varphi),$$

and to say that this is an axiom-schema is to say that every formula that can be obtained from it, by substituting some definite formula for φ and some definite formula for ψ, is an axiom. Clearly, there are infinitely many such formulae, and we count them all as axioms. But despite this prodigality with the axioms, we must also lay down at least one rule of inference, to allow us to make deductions from the axioms. Since the axioms state that certain selected formulae are valid, the sort of rule that we need will be a rule telling us that if such and such formulae are valid, then so also is such and such another formula. By tradition, axiomatic systems almost always adopt here a version of Modus Ponens, which in this context is also called the rule of *detachment*, namely

$$\text{If } \vDash \varphi \text{ and } \vDash \varphi \rightarrow \psi \text{ then } \vDash \psi.$$

An axiomatic system may also adopt other rules, but the general idea is to keep the rules of inference to a minimum, so that it is the axioms rather than the rules which embody the substantial assumptions. Finally, a proof in such a system is just a finite sequence of formulae, each of which is either an axiom or a consequence of preceding formulae by one of the stated rules of inference. It is a proof of the last formula in the sequence. (Note that it follows at once that there is a proof of each axiom, namely the 'sequence' of formulae which consists just of that axiom and nothing else.)

The syntactic turnstile is used in the context

$$\vdash \varphi$$

to mean 'There is a proof of φ' (i.e. in the system currently being considered). Another way of saying the same thing is 'φ is a theorem'. When the proof system is being formally presented, independently of any semantic considerations, we use '$\vdash$' in place of '$\vDash$' to state the axioms and the rules of inference.

EXERCISE

5.1.1. Older axiom systems proceeded not from axiom-schemas but from single axioms, but in addition they adopted as a further rule of proof the principle of uniform substitution for schematic letters (2.5.D and 3.6.I). Now to lay down an axiom-schema is the same as to lay down a single axiom together with a licence to apply the principle of substitution to it. (Why, exactly?) So one could say that the difference between the two approaches is that older systems allowed one to apply substitution at any point in a proof, whereas our approach confines its application to axioms. Consider how one might try to show that the two approaches are equivalent, in the sense that each yields exactly the same theorems. [In effect one has to show that a proof containing a step of substitution applied to a non-axiom can always be replaced by one which eliminates that step, and instead applies substitution only to axioms. The obvious suggestion is: make the same substitution in every formula earlier in the proof. In fact this gives the right answer for formulae that lack quantifiers, but complications can arise when quantifiers are present. Explain. (Recall Exercise 4.9.1.)]

5.2. Axioms for the Truth-Functors

One of the interests in an axiomatic presentation of elementary logic is the economy that can be achieved in the rules and axioms. When combined with the very simple and straightforward structure of proofs in such a system, this can be a considerable help in the investigation of what can and cannot be proved in the system. But economy can be carried too far. For example, it is possible to take a language which contains just one truth-functor, e.g. one of the stroke functors, and to set down just one axiom and one rule for that language, and nevertheless to provide thereby a complete basis for all of truth-functional logic. (See the appendix to this chapter.) But it is horribly difficult to learn to manipulate such a system. We shall, then, aim for a compromise, seeking to economize where it is relatively simple to do so, but not at the cost of losing intelligibility. We may economize on the language, by taking just $\neg,\rightarrow,\forall$ as our logical symbols ($\rightarrow$ because of its connection with Modus Ponens, $\neg$ because it is the natural partner to $\rightarrow$, and $\forall$ for a reason which will become clear in Section 5.6). We shall adopt two axiom-schemas which concern $\rightarrow$ alone, and are deliberately chosen so as to simplify a crucial proof (Section 5.3); one further axiom-schema for $\neg$ (somewhat arbitrarily chosen); and two more axiom-schemas for $\forall$, of which one is very natural and the other is chosen to simplify a crucial proof. Some other possible axiomatizations will be mentioned in the course of the chapter. But I postpone the axioms for $\forall$ to Section 5.6, so that we can begin by confining attention just to the logic of truth-functors.

We shall take, then, a language with $\neg$ and $\rightarrow$ as its only logical symbols. For this language there will be three axiom-schemas, each generating infinitely many axioms, namely

(A1) $\vdash \varphi \rightarrow (\psi \rightarrow \varphi)$.
(A2) $\vdash (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$.
(A3) $\vdash (\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$.

and there will be one rule of inference:

DET: If $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$ then $\vdash \psi$.

It is very easily seen that this system is *sound*, i.e. that every provable formula is valid:

If $\vdash \varphi$ then $\models \varphi$.

In effect we have only to observe that each axiom is valid, and that the one rule of inference preserves validity, and that yields the result at once. To put this argument more fully, we argue by induction on the length of a proof to show that every proof has a valid formula as its last line. Consider, then, any arbitrary proof $\mathcal{P}$, with a formula φ as its last line. The hypothesis of induction is

Every proof shorter than $\mathcal{P}$ has a valid last line,

and we have to show that φ must therefore be a valid formula. Since $\mathcal{P}$ is a proof, we have two cases to consider.

Case (1): φ is an axiom. It is easily checked by the tables that every axiom is valid, and this yields our result at once.

Case (2): φ is a consequence, by detachment, of two earlier lines in $\mathcal{P}$. From the definition of a proof it is clear that any initial segment of a proof is itself a proof, and of course a (proper) initial segment of $\mathcal{P}$ is a proof shorter than $\mathcal{P}$. Hence any line in $\mathcal{P}$, other than the last line, is the last line of some proof shorter than $\mathcal{P}$. So, by the hypothesis of induction, it is valid. But it is easily checked that the rule of detachment, applied to valid formulae, yields only valid formulae as results. Hence φ must be valid.

This completes the argument.

I remark at this point that our system is also *complete*, i.e. that for any formula φ of the language,

$$\text{If} \models \varphi \text{ then } \vdash \varphi.$$

But I postpone a proof of this claim to Section 5.4. Meanwhile, I turn to a different topic, the *independence* of our three axiom-schemas.

Clearly, our axiom-schemas will not be economically chosen if one of the axiom-schemas is superfluous, in that all the axioms generated from it could be obtained as theorems by using only the other axioms. We have to show that this is not the case. In other words we have to show (*a*) that from axiom-schemas (A1) and (A2) together one cannot deduce all the instances of axiom-schema (A3); in particular one cannot deduce the instance

$$(\neg P \rightarrow \neg Q) \rightarrow (Q \rightarrow P).$$

(If one could deduce this instance, then one could also deduce every other. Why? Consider Exercise 5.1.1.) Similarly, we have to show (*b*) that there is an instance of axiom-schema (A2) that cannot be deduced from (A1) and (A3) together; and (*c*) that there is an instance of axiom-schema (A1) that cannot be deduced from axioms (A2) and (A3) together. Now one can show that some formula can be deduced by actually producing the deduction. But how are we to show that a given formula cannot be deduced? The general method is this: one finds some property possessed by every permitted axiom, and preserved by the rule of inference, but not possessed by the formula in question.

The independence of axiom-schema (A3) is easily established in this way. We have observed that, on the standard interpretations of $\neg$ and $\rightarrow$, every axiom is valid and the rule of inference preserves validity. Evidently, this need not be true for non-standard interpretations of these symbols. In particular, consider this non-standard interpretation: $\rightarrow$ is to be interpreted as usual, but $\neg$ is to be interpreted so that $\neg\varphi$ always has the same truth-value as φ. (In other words, we interpret $\neg$ as we standardly interpret $\neg\neg$.) Under this interpretation it is clear that all instances of (A1) and (A2) remain 'valid', and that the rule of inference preserves 'validity', since $\rightarrow$ is not affected. But a typical instance of (A3), such as

$$(\neg P \rightarrow \neg Q) \rightarrow (Q \rightarrow P)$$

is now given the same interpretation as

$$(P \rightarrow Q) \rightarrow (Q \rightarrow P)$$

and so it is not 'valid'. (It takes the value F when $|P| = \text{F}$ and $|Q| = \text{T}$.) It follows that this instance of (A3) cannot be deduced from (A1) and (A2) by the rule of inference. For everything that can be so deduced is not only valid under the standard interpretation of $\rightarrow$ and $\neg$, but also 'valid' under the non-standard interpretation just given.

A slightly different way of putting the same argument is this. Consider a transformation f which transforms each formula of our language into another, by erasing all the negation signs. That is, f is a function from formulae to formulae which obeys these conditions

(1) $f(P_i) = P_i$, for P_i an atomic formula.
(2) $f(\varphi \rightarrow \psi) = (f(\varphi) \rightarrow f(\psi))$.
(3) $f(\neg\varphi) = f(\varphi)$.

Then we argue as follows: if φ is an instance of (A1) or (A2), then $f(\varphi)$ is valid (under the *standard* interpretation of all the signs involved). (This is because $f(\varphi)$ is also an instance of the same axiom-schema.) Also, if $f(\varphi)$ is valid, and $f(\varphi \rightarrow \psi)$ is valid, then $f(\psi)$ is valid too. So it follows that if φ is any formula deducible, by the rule of inference, from (A1) and (A2), then $f(\varphi)$ is valid. But we have seen that if φ is a typical instance of axiom-schema (A3), then $f(\varphi)$ is not valid. It follows that φ cannot be so deduced.

It was relatively easy to show the independence of (A3), because this schema contains a new symbol, $\neg$, that is not present in the other schemata. It is rather more difficult to show the independence of (A1) and (A2). We may begin by looking for some non-standard interpretation of $\rightarrow$, on which (A2) is not counted as 'valid', but (A1) and (A3) are still 'valid', and the rule of inference still preserves 'validity'; or vice versa (A1) is not 'valid' but (A2) and (A3) still are. However, if we confine our attention to the interpretations that can be given in terms of the usual two-valued truth-tables, we shall not find one. (And this remains true even if (A3) is ignored. On the usual two-valued truth-tables, any interpretation of $\rightarrow$ that verifies one of (A1) and (A2) also verifies the other.) We must, then, look to a non-standard interpretation which goes beyond the usual, two-valued, truth-tables.

The usual ploy here is to introduce three-valued tables. These can be thought of in various ways, but for present purposes this approach will be adequate: we shall retain the familiar values T and F, but we shall add another value, which we shall think of as 'between' those two, and which we shall call N (for 'Neither'). (This is intended simply as a helpful prop for thought. But for the formal technique to work it does not matter in the slightest whether the supposed 'values'—T,F,N—can be given an interpretation which makes any kind of sense. Compare the tables given in Exercise 5.2.2(c).) As before, a formula will count as 'valid' iff it always takes the value T, on our tables, whatever the values of its sentence-letters. It follows from this that when φ and ψ both take the value N then $\varphi\to\psi$ must not take the value N (for if it did neither of (A1) and (A2) would be valid). To deal with this point, it is natural to say that even when we have three values to consider we shall preserve the principle that $\varphi\to\psi$ takes the value T whenever φ and ψ take the same value. It is also fairly natural to preserve the principles that $\varphi\to\psi$ takes the value T whenever ψ takes the value T, and whenever φ takes the value F, and that it takes the value F when φ is T and ψ is F. These decisions have already filled in *most* of the places in our new three-valued tables. In fact we have

		ψ		
$\varphi\to\psi$		T	N	F
φ	T	T		F
	N	T	T	
	F	T	T	T

which I abbreviate to

$\to$	T	N	F
T	T		F
N	T	T	
F	T	T	T

(The table on the left is, I hope, self-explanatory, and thus explains the briefer table on the right that I shall use henceforth.) We still have two questions to consider, namely the value of $\varphi\to\psi$ (1) when $|\varphi| = \text{T}$ and $|\psi| = \text{N}$, (2) when $|\varphi| = \text{N}$ and $|\psi| = \text{F}$. There is a further restriction to be observed in case (1), namely that we cannot here have $|\varphi\to\psi| = \text{T}$. For if we do have T here, then the rule of inference will not preserve validity. We may observe also that there is no point in considering tables with T in case (2), for such tables will be equivalent to the standard two-valued tables, with N and F taken to be the *same* value.

With so much by way of initial scene-setting, we now have to resort to tedious experiment. The four tables left for consideration are:

(I) $\to$	T	N	F
T	T	N	F
N	T	T	N
F	T	T	T

(II) $\to$	T	N	F
T	T	F	F
N	T	T	F
F	T	T	T

(III) $\to$	T	N	F
T	T	N	F
N	T	T	F
F	T	T	T

(IV) $\to$	T	N	F
T	T	F	F
N	T	T	N
F	T	T	T

We first try these tables simply on typical instances of (A1) and (A2), namely

(1) $P \to (Q \to P)$
(2) $(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$.

We find that on table I we have (1) valid and (2) invalid (for if $|P| = |Q| = \text{N}$ and $|R| = \text{F}$, then (2) takes the value N); on table II we have (1) invalid and (2) valid (for if $|P| = \text{N}$ and $|Q| = \text{T}$, then (1) takes the value N); on table III both (1) and (2) are valid; on table IV both (1) and (2) are invalid (for if $|P| = \text{N}$ and $|Q| = \text{T}$, then (1) takes the value N, and if $|P| = |Q| = \text{N}$ and $|R| = \text{F}$, then (2) takes the value F). For present purposes, then, the tables of interest are I and II, since the other two do not discriminate between our first two axioms.

We must now add a suitable three-valued table for the negation sign. The natural candidates to consider are these three:

(V) $\neg$	
T	F
N	N
F	T

(VI) $\neg$	
T	F
N	F
F	T

(VII) $\neg$	
T	F
N	T
F	T

It turns out that V will serve our purpose perfectly well, for a typical instance of (A3), namely

(3) $(\neg P \to \neg Q) \to (Q \to P)$,

is valid both on tables I and V and on tables II and V. Thus axiom-schema (A1) is independent of (A2) and (A3), since on tables II and V together both (A2) and (A3) are valid while (A1) is not; and (A2) is independent of (A1) and (A3), since on tables I and V together both (A1) and (A3) are valid while (A2) is not. The argument requires us also to observe that on any of these tables the rule of detachment preserves validity. (I remark, incidentally, that our tables give us another proof that (A3) is independent of (A1) and (A2), since on tables III and VI both (A1) and (A2) are valid while (A3) is not. The same applies to tables III and VII.)

Unfortunately, there is little that one can say by way of advice on finding independence proofs such as these. For the most part, it is a tedious matter of experiment by trial and error. Moreover, there are, in principle, no limits on the complexity of the tables that may be needed: one cannot guarantee that if axioms are independent, then this can be shown by n-valued tables for any specified n, and it may be much more effective to use a different kind of semantics altogether. I give only a very brief indication here of how this might be done. The interpretations for $\to$ that we have been considering have been, in an extended sense, truth-functional. But there is no need to limit attention to such interpretations. For example, suppose that $P \rightarrow Q$ is to be interpreted as 'it is a *necessary* truth that if P then Q'. Then it should be possible to see that axiom-schema (A1) is not correct for this interpretation, while (A3) certainly is correct (given the standard interpretation for $\neg$), and (A2) is somewhat difficult to think about, but might be correct. Moreover, the rule of detachment preserves correctness. So here is a different way of trying to show that (A1) is independent of (A2) and (A3). But some knowledge of modal logic would be required to carry this line of argument through with full rigour.[1]

I end this section with one further application of our three-valued tables. One might have expected that axioms (A1) and (A2) between them would be enough for the deduction of every theorem in which $\rightarrow$ is the sole logical symbol, and that (A3) would have to be used only for theorems containing $\neg$. This is not the case. For a counter-example, consider the thesis known as Peirce's law:

$$((P \rightarrow Q) \rightarrow P) \rightarrow P.$$

As a truth-table check will show, this thesis is valid on the standard interpretation of $\rightarrow$. As will be proved in Section 5.4, our system is complete, so this thesis is provable in it. But it cannot be proved from axiom-schemas (A1) and (A2) alone. For we have noted that whatever can be proved just from those schemas must also be 'valid' on the three-valued table III, but Peirce's law is not. On those tables, it has the value N when $|P| = \mathrm{N}$ and $|Q| = \mathrm{F}$. I shall come back to the significance of this point on several occasions hereafter.

EXERCISES

5.2.1. Prove the assertion (p. 196) that any interpretation of $\rightarrow$ on the usual two-valued truth-tables will verify either both or neither of axiom-schemas (A1) and (A2). [Of course, this can be done by tediously trying each of the sixteen possible interpretations in turn. But see if you can find a shorter method of argument.]

5.2.2. As will be shown in Section 5.4, in place of our axiom-schema (A3) we could have used this alternative (A3′) instead:

(A3′): $(\neg\varphi \rightarrow \psi) \rightarrow ((\neg\varphi \rightarrow \neg\psi) \rightarrow \varphi)$.

[1] The fact is that, under the suggested interpretation for $\rightarrow$, (A2) and (A3) are valid in the modal logic S4, and so is anything that can be deduced from them by the rule of detachment, while (A1) is not valid in any modal logic.

(*a*) Show that (A3′) is independent of (A1) and (A2). [The same argument as works for (A3) will also work for (A3′).]
(*b*) Show that (A1) is independent of (A2) and (A3′). [Use table VII in place of table V.]
(*c*) Show that no combination of tables I–VII will demonstrate that (A2) is independent of (A1) and (A3′), but that the following unexpected tables will do the trick:

$\rightarrow$	T	N	F
T	T	F	F
N	T	F	T
F	T	T	T

$\neg$	
T	N
N	N
F	N

5.3. The Deduction Theorem

So far, we have established various results about what *cannot* be proved from this or that set of axioms, but we have not shown that anything *can* be proved. This will now be remedied. As we have observed, the most straightforward way of showing that something can be proved is by actually giving a proof of it, and we will begin with an example. Here is a proof of the simple theorem $P\rightarrow P$.

$\vdash P\rightarrow P$

1.	$(P\rightarrow((P\rightarrow P)\rightarrow P)) \rightarrow ((P\rightarrow(P\rightarrow P)) \rightarrow (P\rightarrow P))$	A2
2.	$P\rightarrow((P\rightarrow P)\rightarrow P)$	A1
3.	$P\rightarrow(P\rightarrow P)$	A1
4.	$(P\rightarrow(P\rightarrow P)) \rightarrow (P\rightarrow P)$	1,2
5.	$P\rightarrow P$	3,4

On the right we have noted the justification for each line of the proof. Thus the first line is an instance of axiom-schema (A2) and the second and third are instances of axiom-schema (A1). The fourth line follows by the rule of detachment from lines (1) and (2) earlier, and the fifth line follows similarly from lines (3) and (4). It is standard practice always to furnish a justification for each line of a proof, so that it can easily be checked that indeed it is a proof. Now let us note two things about this proof.

First, it is obvious that by the same proof as we have used to prove $P\rightarrow P$ one could also prove any substitution-instance of that theorem, e.g.

$Q \rightarrow Q$
$(P \rightarrow Q) \rightarrow (P \rightarrow Q)$
$\neg(P \rightarrow Q) \rightarrow \neg(P \rightarrow Q)$
etc.

All one has to do to find a proof of the substitution-instance is to make the same substitutions all the way through the original proof. It is clear that this argument holds quite generally, so we can say that in our axiomatic system uniform substitution for schematic letters preserves provability, i.e.

A substitution-instance of a theorem is itself a theorem.

To save having to cite this principle explicitly every time that we wish to use it, in future we shall very seldom cite actual theorems of the system, or give actual proofs of them, but will cite theorem-*schemas* (like our axiom-schemas), and give proof-schemas to establish them. So the above proof would be given with the schematic φ in place of the definite formula P throughout, and would be taken as establishing the theorem-schema

Lemma. $\vdash \varphi \rightarrow \varphi$.

(But, to avoid prolixity, we shall often in practice refer to theorem-schemas simply as theorems, and to axiom-schemas as axioms).

A second point to note about the proof just given is that it is remarkably roundabout. When seeking for a proof of the simple theorem $P \rightarrow P$, how would one know that anything as complicated as line (1) would be needed? What principles are there that can guide one to the right axioms to use in the first place? Well, the answer is that the problem of finding proofs can be very much simplified if we begin by looking for a different *kind* of proof altogether, namely a *proof from assumptions*.

We shall use '$\Gamma \vdash \varphi$' to mean 'there is a proof of φ from the set of assumptions Γ', and we define this as short for 'there is a finite sequence of formulae such that each of them is either an axiom, or a member of Γ, or a consequence of previous formulae by one of the specified rules of inference; and the last formula in the sequence is φ'. It is worth noting at once that from this definition there follow without more ado the three 'structural' principles of Assumptions, Thinning, and Cutting, namely

ASS: $\varphi \vdash \varphi$
THIN: If $\Gamma \vdash \varphi$ then $\Gamma, \psi \vdash \varphi$
CUT: If $\Gamma \vdash \varphi$ and $\phi, \Delta \vdash \psi$ then $\Gamma, \Delta \vdash \psi$.

These follow independently of how the rules of inference are specified. For example, to verify the principle of Assumptions we have only to note that the sequence consisting of the single formula φ and nothing else is, by the definition, a proof of φ from the assumption φ. The other two principles are proved equally simply. But let us now come to the question of specifying rules of inference.

Rules of inference for use in proofs from assumptions need not be the same as those specified for axiomatic proofs, but they should have the axiomatic rules as a special case, namely the case where there are no assumptions. Thus at present our axioms are (A1)–(A3) as specified in the last section, and as our rule of inference we now take Modus Ponens in its most general form, which is usually written as

$$\varphi,\varphi\rightarrow\psi \vdash \psi.$$

But since we are licensed to apply this rule within proofs from assumptions, a more precise formulation is

$$\text{If } \Gamma\vdash\varphi \text{ and } \Gamma\vdash\varphi\rightarrow\psi \text{ then } \Gamma\vdash\psi.$$

It is therefore not the *same* rule as the rule of detachment that we began with, but a more general rule. For the rule of detachment is the special case of this rule in which Γ is null, i.e. in which there are no assumptions. Consequently, proofs in the axiomatic system that we began with are special cases of proofs from assumptions in general, namely the cases in which there are no assumptions.

Now, it is very much easier to look for proofs from assumptions than it is to look for proper axiomatic proofs. Yet also, every proof from assumptions can be transformed into a proper axiomatic proof in this sense: a proof of ψ from the assumptions $\varphi_1,...,\varphi_n$, showing that

$$\varphi_1,...,\varphi_n \vdash \psi$$

can be transformed to a proof showing that

$$\varphi_1,...,\varphi_{n-1} \vdash \varphi_n\rightarrow\psi,$$

and this in turn can be transformed to show that

$$\varphi_1,...,\varphi_{n-2} \vdash \varphi_{n-1}\rightarrow(\varphi_n\rightarrow\psi)$$

and by repeating such transformations as often as necessary we eventually get a proof from no assumptions showing that

$$\vdash \varphi_1\rightarrow(\varphi_2\rightarrow(...(\varphi_n\rightarrow\psi)..)).$$

This last is called the *conditionalization* of the sequent with which we began. To establish the claim, we prove what is called the *Deduction Theorem*, which states that

If $\Gamma,\varphi \vdash \psi$ then $\Gamma \vdash \varphi \to \psi$.

(Compare 2.5.H.)

The argument is by induction on the length of the proof showing that $\Gamma,\varphi \vdash \psi$. So let $\mathcal{P}$ be such a proof. The hypothesis of induction is

> Any proof shorter than $\mathcal{P}$ showing that $\Gamma,\varphi \vdash \chi$ can be transformed into a proof showing that $\Gamma \vdash \varphi \to \chi$, for any formula χ.

We have three cases to consider, according to the three possible justifications for the line ψ in the proof.

Case (1): ψ is an axiom. Then $\psi \to (\varphi \to \psi)$ is also an axiom. So by Modus Ponens $\varphi \to \psi$ is provable on no assumptions. So we have (by Thinning) a proof that $\Gamma \vdash \varphi \to \psi$.

Case (2): ψ is an assumption.

Subcase (*a*): ψ is φ. Then $\varphi \to \psi$ is $\varphi \to \varphi$, and so is provable on no assumptions. (Lemma, proved above.) So we have (by Thinning) a proof that $\Gamma \vdash \varphi \to \psi$.

Subcase (*b*): ψ is in Γ. Then there is a proof (by Assumptions, and Thinning) that $\Gamma \vdash \psi$. Add to this proof the lines $\psi \to (\varphi \to \psi)$ and $\varphi \to \psi$. The first adds an axiom, and the second a consequence by Modus Ponens of two previous lines. So the result is a proof showing that $\Gamma \vdash \varphi \to \psi$.

Case (3): ψ is a consequence by Modus Ponens of two previous lines χ and $\chi \to \psi$. Then by inductive hypothesis there are proofs showing that $\Gamma \vdash \varphi \to \chi$ and $\Gamma \vdash \varphi \to (\chi \to \psi)$. Put these proofs together and add the lines $(\varphi \to (\chi \to \psi)) \to ((\varphi \to \chi) \to (\varphi \to \psi))$, $(\varphi \to \chi) \to (\varphi \to \psi)$, and $\varphi \to \psi$. The first adds an axiom, and the second and third add consequences of previous lines by Modus Ponens. So the result is a proof showing that $\Gamma \vdash \varphi \to \psi$.

This completes the argument. It may be noted that axioms (A1) and (A2), together with their consequence $\varphi \to \varphi$, are exactly the premisses needed to push the argument through. The axioms were chosen precisely for this purpose.

Let us illustrate the use of the deduction theorem by proving, with its help, some simple sequents. A nice easy example is

$\varphi\to\psi, \psi\to\chi \vdash \varphi\to\chi$

1.	$\varphi\to\psi$	ASS
2.	$\psi\to\chi$	ASS
3.	φ	~~ASS~~(6)
4.	ψ	1,3,MP
5.	χ	2,4,MP
6.	$\varphi\to\chi$	3–5,Deduction theorem

The citation 'ASS' as a justification is short for 'Assumption'. We begin the proof by writing as the first two lines the assumptions in the sequent we are trying to prove. Then in line (3) we introduce an *auxiliary* assumption, which will be *discharged* (i.e. will no longer be an assumption) by the time we have got to the end of this proof. When we introduce it, we simply write 'ASS' in justification; the line through 'ASS' will be added later, in fact when we come to line (6), at which the assumption is discharged. Lines (4) and (5) are then simple applications of Modus Ponens (abbreviated to MP) to previous lines. By the time that we have reached line (5) what we have shown is

$$\varphi\to\psi, \psi\to\chi, \varphi \vdash \chi.$$

At this stage we cite the deduction theorem, which tells us that our proof so far may be transformed into another proof which shows that

$$\varphi\to\psi, \psi\to\chi \vdash \varphi\to\chi.$$

We do not write that other proof out, but simply rely on the fact that that other proof does exist, as the deduction theorem has shown. Consequently, at line (6) we write the desired conclusion $\varphi\to\chi$, we discharge the assumption φ by putting a line through its original justification 'ASS', and we add the tag (6) to show at what point this assumption was discharged. Finally, in justification of line (6) we cite the line at which φ was introduced as an assumption, the line at which the conclusion χ was obtained, and the deduction theorem itself. (In future I shall abbreviate the citation 'Deduction theorem' simply to 'D'. In many books this important principle is called 'the rule of Conditional Proof', abbreviated to 'CP'. We shall meet another name for it in the next chapter.)

I remarked that when we apply the deduction theorem we do not write out a further proof, but just rely on the fact that it exists. Of course, the proof of the deduction theorem shows us how to write out such a proof if we wish to, but the proof found by that method is usually quite unnecessarily roundabout. I illustrate by applying the method to the proof just given, adding to each original assumption two extra lines, showing that it can be prefaced by $\varphi\to$, and replacing other lines by proofs which show that they too can be prefaced by $\varphi\to$. The result is this:

$\varphi\rightarrow\psi, \psi\rightarrow\chi \vdash \varphi\rightarrow\chi$

1.	$\varphi\rightarrow\psi$	ASS
1*a*.	$(\varphi\rightarrow\psi) \rightarrow (\varphi\rightarrow(\varphi\rightarrow\psi))$	A1
1*b*.	$\varphi\rightarrow(\varphi\rightarrow\psi)$	1,1*a*,MP
2.	$\psi\rightarrow\chi$	ASS
2*a*.	$(\psi\rightarrow\chi) \rightarrow (\varphi\rightarrow(\psi\rightarrow\chi))$	A1
2*b*.	$\varphi\rightarrow(\psi\rightarrow\chi)$	2,2*a*,MP
3*a*.	$\varphi\rightarrow\varphi$	Lemma
4*a*.	$(\varphi\rightarrow(\varphi\rightarrow\psi)) \rightarrow ((\varphi\rightarrow\varphi)\rightarrow(\varphi\rightarrow\psi))$	A2
4*b*.	$(\varphi\rightarrow\varphi)\rightarrow(\varphi\rightarrow\psi)$	1*b*,4*a*,MP
4*c*.	$\varphi\rightarrow\psi$	3*a*,4*b*,MP
5*a*.	$(\varphi\rightarrow(\psi\rightarrow\chi)) \rightarrow ((\varphi\rightarrow\psi)\rightarrow(\varphi\rightarrow\chi))$	A2
5*b*.	$(\varphi\rightarrow\psi) \rightarrow (\varphi\rightarrow\chi)$	2*b*,5*a*,MP
5*c*.	$\varphi\rightarrow\chi$	4*c*,5*b*,MP

It is to be noted that lines (1*a*), (1*b*), (3*a*), (4*a*), (4*b*) were all added to the original proof in order to obtain $\varphi\rightarrow\psi$ in line (4*c*). This was all quite unnecessary, since $\varphi\rightarrow\psi$ is one of the assumptions to the proof anyway. So we may obviously simplify the proof in this way:

$\varphi\rightarrow\psi, \psi\rightarrow\chi \vdash \varphi\rightarrow\chi$

1.	$\varphi\rightarrow\psi$	ASS
2.	$\psi\rightarrow\chi$	ASS
3.	$(\psi\rightarrow\chi) \rightarrow (\varphi\rightarrow(\psi\rightarrow\chi))$	A1
4.	$\varphi\rightarrow(\psi\rightarrow\chi)$	2,3,MP
5.	$(\varphi\rightarrow(\psi\rightarrow\chi)) \rightarrow ((\varphi\rightarrow\psi)\rightarrow(\varphi\rightarrow\chi))$	A2
6.	$(\varphi\rightarrow\psi)\rightarrow(\varphi\rightarrow\chi)$	4,5,MP
7.	$\varphi\rightarrow\chi$	1,6,MP

This is the simplest proof of the result, *not* using the deduction theorem, that I am aware of. It is only one line longer than our original proof, which did use the deduction theorem. But it is much more difficult to find.

The point can be made yet more obvious if we introduce a further step of complication, and consider the sequent

$$\varphi\rightarrow\psi \vdash (\psi\rightarrow\chi)\rightarrow(\varphi\rightarrow\chi).$$

If we may make use of the deduction theorem, then the proof of this is extremely simple. We repeat the original proof given on p. 204 and add one further use of the deduction theorem at the end, discharging assumption (2). This assures us that there is a proof of the sequent which does not use the deduction theorem. But when one tries to find such a proof, it turns out to be very complicated indeed, as you are invited to discover. One can, of course, continue the lesson yet further by considering proofs, with and without the deduction theorem, of the sequent

$$\vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)).$$

Use of the deduction theorem greatly simplifies the search for proof (or, more strictly, the task of showing that there is a proof). Once this theorem is available, there is usually no difficulty in finding a proof (or, a proof that there is a proof) of any sequent whose proof depends simply on axioms (A1) and (A2). Some simple examples are suggested in the exercises to this section. I postpone to the next section the use of axiom (A3). Meanwhile, I bring this section to an end with some further reflections upon the content of our axioms (A1) and (A2).

We have seen that from axioms (A1) and (A2), together with the rule Modus Ponens, one can prove the deduction theorem:

If $\Gamma,\varphi \vdash \psi$ then $\Gamma \vdash \varphi \to \psi$.

The converse is also true. Given the deduction theorem (as stated here), and Modus Ponens, one can prove (A1) and (A2). Here is a proof of (A1):

$\vdash \varphi \to (\psi \to \varphi)$

1.	ψ	~~ASS~~(3)
2.	φ	~~ASS~~(4)
3.	$\psi \to \varphi$	1–2,D
4.	$\varphi \to (\psi \to \varphi)$	2–3,D

When the proof is set out like this, it may seem to be something of a cheat. In lines (1) and (2) we introduce two assumptions. Then in line (3) we infer that if the first is true, so is the second, and discharge the first, so that this conclusion depends only on the second. But, of course, we have not, in any intuitive sense, *deduced* the second assumption from the first, and so this step is certainly unexpected. It is, however, entirely in accordance with the deduction theorem as we have stated it. For the two-line proof which consists of first ψ and then φ, with each line entered as an assumption, is a proof showing that $\psi,\varphi \vdash \varphi$. (Compare the remark earlier on $\varphi \vdash \varphi$.) Since the order of the premisses makes no difference, we can also say that it is a proof showing that $\varphi, \psi \vdash \varphi$. To this we apply the deduction theorem, as stated, to infer that $\varphi \vdash \psi \to \varphi$, and that is exactly what line (3) records. The further step of the deduction theorem in line (4) is then completely straightforward.

This shows that axiom (A1) does indeed follow from the deduction theorem, as stated. Axiom (A2) follows also, if we allow Modus Ponens to be used too, as this proof shows:

$\vdash (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

1.	$\varphi \to (\psi \to \chi)$	~~ASS~~(9)
2.	$\varphi \to \psi$	~~ASS~~(8)
3.	φ	~~ASS~~(7)
4.	$\psi \to \chi$	1,3,MP
5.	ψ	2,3,MP
6.	χ	4,5,MP
7.	$\varphi \to \chi$	3–6,D
8.	$(\varphi \to \psi) \to (\varphi \to \chi)$	2–7,D
9.	$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$	1–8,D

In the presence of Modus Ponens, then, (A1) and (A2) are together equivalent to the deduction theorem. It may also be noted that Modus Ponens is itself equivalent to the converse of the deduction theorem, namely

If $\Gamma \vdash \varphi \to \psi$ then $\Gamma, \varphi \vdash \psi$.

To see that Modus Ponens follows from this principle we have only to consider the special case in which Γ is the formula $\varphi \to \psi$. Then the left-hand side is simply a case of the principle of Assumptions, so we may infer the correctness of the right-hand side, which is Modus Ponens. As for the argument in the other direction, if we suppose that $\Gamma \vdash \varphi \to \psi$, and we also assume that $\varphi \to \psi, \varphi \vdash \psi$, then by an application of CUT it follows at once that $\Gamma, \varphi \vdash \psi$.

We may conclude that the assumptions about $\to$ that are stated in axioms (A1) and (A2) of the axiomatic system, together with the rule of inference Modus Ponens, are actually equivalent to this assumption (namely 2.5.H):

$\Gamma \vdash \varphi \to \psi$ iff $\Gamma, \varphi \vdash \psi$.

Whatever follows from the one will therefore follow from the other also. But we have observed (at the end of Section 5.2) that not all truths about $\to$ do follow from the assumptions in question. Even when our attention is focused just on $\to$, we cannot ignore the effect of our axiom (A3).

EXERCISES

5.3.1. Use the deduction theorem to prove the following:

(*a*) $\vdash \varphi \to \varphi$.
(*b*) $\varphi \to (\varphi \to \psi) \vdash \varphi \to \psi$.
(*c*) $\varphi \to (\psi \to \chi) \vdash \psi \to (\varphi \to \chi)$.
(*d*) $(\varphi \to \psi) \to \chi \vdash \psi \to \chi$.
(*e*) $((\varphi \to \psi) \to \psi) \to \chi \vdash \varphi \to \chi$.

5.3.2. Find a proof, not using the deduction theorem, of the sequent

$$\varphi \rightarrow \psi \vdash (\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi).$$

[Method: begin with the seven-line proof on p. 205, adding to it a use of the deduction theorem as a final step. Then eliminate that use by the method employed in proving the deduction theorem, and simplify the result by omitting superfluous detours. This should yield a proof of thirteen lines, using three instances of each axiom-schema.]

5.3.3. To axioms (A1) and (A2) add a further axiom:

(P) $\vdash ((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi$.

(*a*) Observe that the proof of the deduction theorem is not affected by adding a new axiom.

(*b*) Using the new axiom (and the deduction theorem, and Modus Ponens) prove the sequent

$$(\varphi \rightarrow \psi) \rightarrow \psi \vdash (\psi \rightarrow \varphi) \rightarrow \varphi.$$

(*c*) Show that this sequent is not provable from (A1) and (A2) alone. [Hint: recall the last paragraph of Section 5.2.]

5.4. **Some Laws of Negation**

Our axiom (A3) for negation was somewhat arbitrarily chosen. There are many other useful and important laws for negation that might perfectly well be used in its place. To begin with, we may note that there are four laws which are together known as the laws of *contraposition*, namely

(i) $\varphi \rightarrow \psi \vDash \neg\psi \rightarrow \neg\varphi$.
(ii) $\varphi \rightarrow \neg\psi \vDash \psi \rightarrow \neg\varphi$.
(iii) $\neg\varphi \rightarrow \psi \vDash \neg\psi \rightarrow \varphi$.
(iv) $\neg\varphi \rightarrow \neg\psi \vDash \psi \rightarrow \varphi$.

It is easily seen that any of these can be deduced from any of the others, given in addition the two laws of double negation, namely

$$\varphi \vDash \neg\neg\varphi.$$
$$\neg\neg\varphi \vDash \varphi.$$

Now in fact our axiom (A3) corresponds to the fourth law of contraposition above, and—as we shall see—both the laws of double negation can be deduced from this. But in place of (A3) we could have had an axiom corresponding to the first law of contraposition, together with two further axioms corresponding to the two laws of double negation. Or we could have had axioms corresponding to the second law of contraposition plus the second law of double negation; or the third law of contraposition plus the first law of double negation. For in either case we should have been able to deduce from these the axiom that we do have, as may easily be checked. But on the other hand there was no need to start from any version of the laws of contraposition, as I now demonstrate. I first give a series of deductions from the axiom (A3) that we have in fact adopted, but I then point out how this series shows that quite different starting-points could have been adopted.

The deductions will assume as background both the rule Modus Ponens (cited as MP) and the deduction theorem (cited as D). In view of my remarks at the end of the last section, I shall count the sequents

$$\vdash \varphi \rightarrow \varphi$$
$$\varphi \vdash \psi \rightarrow \varphi$$

as following from the deduction theorem alone. I shall also suppose that our axiom (A3) may be cited without more ado in the form

$$\neg\varphi \rightarrow \neg\psi \vdash \psi \rightarrow \varphi.$$

With so much by way of preliminaries, let us now proceed to the deductions.

(T1) $\varphi, \neg\varphi \vdash \psi$ (EFQ)

1.	φ	ASS
2.	$\neg\varphi$	ASS
3.	$\neg\psi \rightarrow \neg\varphi$	2,D
4.	$\varphi \rightarrow \psi$	3,A3
5.	ψ	1,4,MP

I label the sequent proved '(T1)', short for 'theorem 1'. I also put in brackets the label 'EFQ', short for *ex falso quodlibet*, which is the usual name for this law.[2] The proof is, I think, perfectly straightforward. So let us proceed.

(T2) $\neg\varphi \rightarrow \varphi \vdash \varphi$ (CM*)

1.	$\neg\varphi \rightarrow \varphi$	ASS
2.	$\neg\varphi$	~~ASS~~(5)
3.	φ	1,2,MP
4.	$\neg(\neg\varphi \rightarrow \varphi)$	2,3,T1
5.	$\neg\varphi \rightarrow \neg(\neg\varphi \rightarrow \varphi)$	2–4,D
6.	$(\neg\varphi \rightarrow \varphi) \rightarrow \varphi$	5,A3
7.	φ	1,6,MP

[2] The label is inaccurate; it means 'from what is false there follows anything you like', but it should say 'from a contradiction', not 'from what is false'.

The label 'CM' stands for *consequentia mirabilis*; I have added an asterisk to distinguish this version from the perhaps more usual version proved as (T4) below. The proof is perhaps rather unexpected; in lines (2) and (3) we have reached a contradiction, so we may employ the result already established as (T1) to infer any formula that we like. We cunningly choose the negation of our first assumption. We then apply the deduction theorem, discharging our second assumption, and this allows us to bring the axiom to bear, so yielding the desired conclusion. Two points should be noted about the use of the result (T1) *within* the proof of (T2). First, we have taken a substitution-instance of (T1) as first stated, writing $\neg(\neg\varphi\rightarrow\varphi)$ in place of ψ. This is a perfectly legitimate procedure, as I have already remarked (p. 201). Second, since (T1) is not itself one of the initially stated rules of inference, the sequence of lines (1)–(4) does not satisfy the original definition of a proof from assumptions (p. 201). It should be regarded, rather, as a proof that there is a proof in the original sense, namely one got by inserting the original proof of (T1) into the proof of (T2) at this point. So, if the proof of (T2) were to be written out more fully, its first four lines would be replaced by

1. $\neg\varphi\rightarrow\varphi$	ASS
2. $\neg\varphi$	~~ASS~~(5)
3. φ	1,2,MP
4*a*. $\neg\neg(\neg\varphi\rightarrow\varphi)\rightarrow\neg\varphi$	2,D
4*b*. $\varphi\rightarrow\neg(\neg\varphi\rightarrow\varphi)$	4*a*,A3
4*c*. $\neg(\neg\varphi\rightarrow\varphi)$	3,4*b*,MP

The citation of one result already proved, within the proof of a further result, is therefore—like citations of the deduction theorem—an indication that there is a proof, but not itself part of that proof. When we are working from a small number of initial axioms, use of this technique is in practice unavoidable. It is also a convenient way of showing what can be proved from what; in the present case, it may be noted that axiom (A3) will never be cited again in this series of deductions. All the further results can be obtained just from (T1) and (T2) as 'basic premisses'.

(T3) $\neg\neg\varphi\vdash\varphi$ (DNE)

1. $\neg\neg\varphi$	ASS
2. $\neg\varphi$	~~ASS~~(4)
3. φ	1,2,T1
4. $\neg\varphi\rightarrow\varphi$	2–3,D
5. φ	4,T2

('DN' abbreviates 'double negation' and 'DNE' is 'double negation elimination'.)

(T4) $\varphi\rightarrow\neg\varphi\vdash\neg\varphi$ (CM)

1.	$\varphi\rightarrow\neg\varphi$	ASS
2.	$\neg\neg\varphi$	~~ASS~~(5)
3.	φ	2,T3
4.	$\neg\varphi$	1,2,MP
5.	$\neg\neg\varphi\rightarrow\neg\varphi$	2–4,D
6.	$\neg\varphi$	5,T2

(T5) $\varphi\rightarrow\psi, \varphi\rightarrow\neg\psi\vdash\neg\varphi$ (RAA)

1.	$\varphi\rightarrow\psi$	ASS
2.	$\varphi\rightarrow\neg\psi$	ASS
3.	φ	~~ASS~~(7)
4.	ψ	1,3,MP
5.	$\neg\psi$	2,3,MP
6.	$\neg\varphi$	4,5,T1
7.	$\varphi\rightarrow\neg\varphi$	3–6,D
8.	$\neg\varphi$	7,T4

'RAA' abbreviates '*reductio ad absurdum*'. (For a strengthened form of RAA, as CM* is stronger than CM, see Exercise 5.4.2.) Note here that (T1), (T2), and (T4) will not be used again in the following proofs, which depend only on (T3) and (T5).

(T6) $\varphi\rightarrow\neg\psi\vdash\psi\rightarrow\neg\varphi$ (CON(ii))

1.	$\varphi\rightarrow\neg\psi$	ASS
2.	ψ	~~ASS~~(5)
3.	$\varphi\rightarrow\psi$	2,D
4.	$\neg\varphi$	1,3,T5
5.	$\psi\rightarrow\neg\varphi$	2–4,D

('CON' abbreviates 'contraposition')

(T7) $\varphi\vdash\neg\neg\varphi$ (DNI)

1.	φ	ASS
2.	$\neg\varphi\rightarrow\neg\varphi$	D
3.	$\varphi\rightarrow\neg\neg\varphi$	2,T6
4.	$\neg\neg\varphi$	1,3,MP

('DNI' abbreviates 'double negation introduction'.)

(T8) $\varphi \to \psi \vdash \neg\psi \to \neg\varphi$ (CON(i))

1.	$\varphi \to \psi$	ASS
2.	φ	~~ASS~~(5)
3.	ψ	1,2,MP
4.	$\neg\neg\psi$	3,T7
5.	$\varphi \to \neg\neg\psi$	2–4,D
6.	$\neg\psi \to \neg\varphi$	5,T6

(T9) $\neg\varphi \to \psi \vdash \neg\psi \to \varphi$ (CON(iii))

1.	$\neg\varphi \to \psi$	ASS
2.	$\neg\psi$	~~ASS~~(6)
3.	$\neg\varphi \to \neg\psi$	2,D
4.	$\neg\neg\varphi$	1,3,T5
5.	φ	4,T3
6.	$\neg\psi \to \varphi$	2–5,D

(T10) $\varphi \to \psi, \neg\varphi \to \psi \vdash \psi$ (TND)

1.	$\varphi \to \psi$	ASS
2.	$\neg\varphi \to \psi$	ASS
3.	$\neg\psi \to \neg\varphi$	1,T8
4.	$\neg\psi \to \neg\neg\varphi$	2,T8
5.	$\neg\neg\psi$	3,4,T5
6.	ψ	5,T3

'TND' abbreviates 'tertium non datur', which is another name for the law of excluded middle (LEM). (Literally 'TND' means 'A third (possibility) is not given'.) Properly speaking, LEM is $\vdash \varphi \vee \neg\varphi$, and so requires $\vee$ in its formulation, and is not yet available. What is here named TND is perhaps best viewed as a *consequence* of LEM, since it says in effect that if a conclusion ψ can be got both from φ as assumption and from $\neg\varphi$ as assumption then it must be true, which, of course, is because those two assumptions between them exhaust the possibilities. Given standard rules for $\vee$, as in Section 5.7 below, one can very swiftly deduce TND as stated here from LEM as properly formulated.

I now introduce some reverse deductions. First, CM, which was used in the proof of RAA, is in fact a special case of RAA, as the following proof shows:

CM: $\varphi\to\neg\varphi \vdash \neg\varphi$

1. $\varphi\to\neg\varphi$ ASS
2. $\varphi\to\varphi$ D
3. $\neg\varphi$ 1,2,RAA(=T5)

In a similar way CM* is a special case of TND:

CM*: $\neg\varphi\to\varphi \vdash \varphi$

1. $\neg\varphi\to\varphi$ ASS
2. $\varphi\to\varphi$ D
3. φ 1,2,TND(=T10)

This latter is rather more significant, since it shows that whatever can be deduced from CM* (=T2) can also be deduced from TND (=T10), and several important theses were deduced from CM*. Another quite significant reverse deduction is that EFQ, which was our first theorem, could have been obtained instead from the third law of contraposition, which was (T9):

EFQ: $\varphi,\neg\varphi \vdash \psi$

1. φ ASS
2. $\neg\varphi$ ASS
3. $\neg\psi\to\varphi$ 1,D
4. $\neg\varphi\to\psi$ 3,CON(iii) (=T9)
5. ψ 2,4,MP.

I now add two of the points made at the outset of this section, that from CON(ii) with DNE, or from CON(iii) with DNI, it is possible to recover the original axiom (A3). Here are proofs.

(A3) $\neg\varphi\to\neg\psi \vdash \psi\to\varphi$

1. $\neg\varphi\to\neg\psi$ ASS
2. $\psi\to\neg\neg\varphi$ 1,CON(ii) (=T6)
3. ψ ~~ASS~~(6)
4. $\neg\neg\varphi$ 2,3,MP
5. φ 4,DNE(=T3)
6. $\psi\to\varphi$ 3–5,D

(A3) $\neg\varphi\to\neg\psi \vdash \psi\to\varphi$

1. $\neg\varphi\to\neg\psi$ ASS
2. $\neg\neg\psi\to\varphi$ 1,CON(iii) (=T9)
3. ψ ~~ASS~~(6)
4. $\neg\neg\psi$ 3,DNI(=T7)

5. φ 2,4,MP
6. $\psi \to \varphi$ 3–5,D

The results of all these deductions may conveniently be surveyed in the following diagram (where the arrows indicate that the sequent at the pointed end of the arrows may be proved from the sequents at the other end of those arrows, assuming the rules MP and D as background rules).

From this diagram one can at once read off that each of the following sets of basic axioms would be equivalent to the single axiom (A3):

(*a*) TND(=T10) + EFQ(=T1)

$\varphi \to \psi, \neg\varphi \to \psi \vdash \psi$
$\varphi, \neg\varphi \vdash \psi$

(*b*) CM*(=T2) + EFQ(=T1)

$\neg\varphi \to \varphi \vdash \varphi$
$\varphi, \neg\varphi \vdash \psi$

(*c*) TND(=T10) + RAA(=T5)

$\varphi \to \psi, \neg\varphi \to \psi \vdash \psi$
$\varphi \to \psi, \varphi \to \neg\psi \vdash \neg\varphi$

(*d*) DNE(=T3) + RAA(=T5)

$\neg\neg\varphi \vdash \varphi$
$\varphi \to \psi, \varphi \to \neg\psi \vdash \neg\varphi$

(*e*) DNE(=T3) + CON(ii)(=T6)

$\neg\neg\varphi \vdash \varphi$
$\varphi \to \neg\psi \vdash \psi \to \neg\varphi$

(*f*) CON(iii)(=T9) + DNI(=T7)

$\neg\varphi \to \psi \vdash \neg\psi \to \varphi$
$\varphi \vdash \neg\neg\varphi$

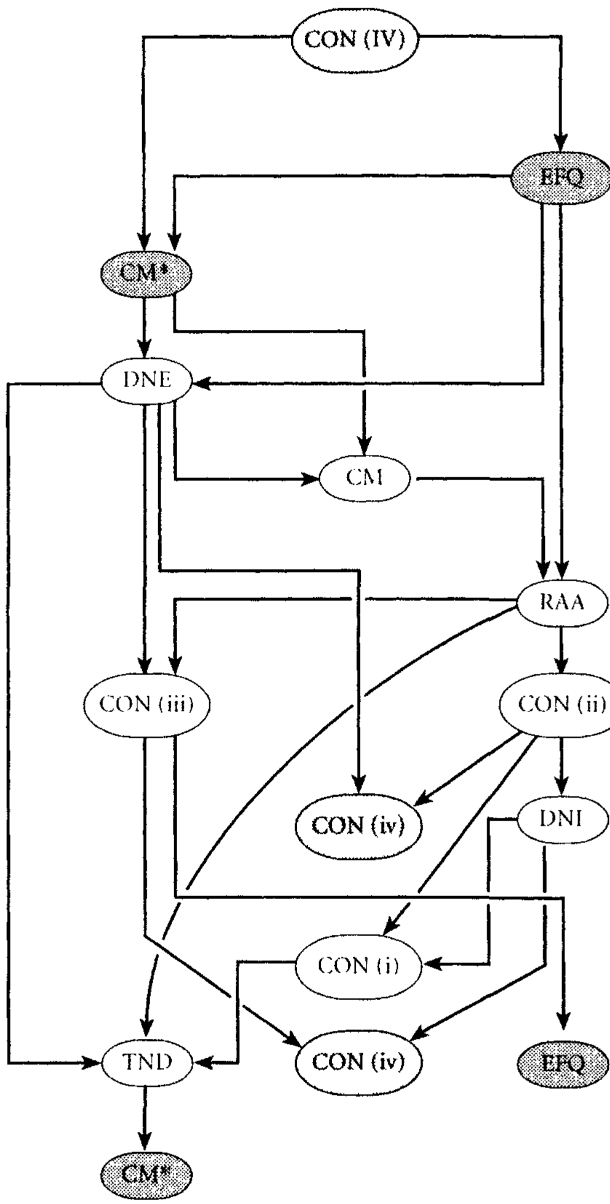
(*g*) CON(iii)(=T9) + CM*(=T2)

$\neg\varphi \to \psi \vdash \neg\psi \to \varphi$
$\neg\varphi \to \varphi \vdash \varphi$

(*h*) CON(iii)(=T9) + CON(ii)(=T6)

$\neg\varphi \to \psi \vdash \neg\psi \to \varphi$
$\varphi \to \neg\psi \vdash \psi \to \neg\varphi$

There are yet other combinations which are again equivalent, not only those that can be read off the diagram as it stands, but also some that would



be revealed by complicating the diagram still further. This matter is further explored in the exercises.

EXERCISES

5.4.1. Many books choose DNE and RAA as their basic assumptions on negation. The diagram shows that all the theses we have considered can be obtained from this basis, but often the route suggested is rather indirect.

(*a*) Using DNE and RAA as basic assumptions, give direct proofs of EFQ, CM*, and CON(i), not relying on any other thesis.
(*b*) Using the same assumptions, give a proof of Peirce's law

$$\vdash ((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi.$$

5.4.2. Show that the single thesis RAA*, namely

$$\vdash (\neg\varphi \rightarrow \psi) \rightarrow ((\neg\varphi \rightarrow \neg\psi) \rightarrow \varphi),$$

is equivalent to the single axiom (A3) that we have adopted. [It is easy to deduce RAA* from RAA and DNE. For the converse deduction, perhaps the simplest plan is to deduce both EFQ and CM* directly from RAA*. The argument can then be concluded by relying on the results presented on the diagram.]

5.4.3. Suppose that we define negation by putting

$$\neg\varphi \quad \text{for} \quad \varphi \rightarrow \perp.$$

(*a*) Without adding any extra assumptions about $\perp$, show that the following theses are immediate consequences of the definition: CM, RAA, CON(ii), DNI, CON(i).
(*b*) Adding a further assumption about $\perp$, namely

$$\vdash \perp \rightarrow \psi,$$

show that EFQ may then be counted a consequence of the definition.
(*c*) Independently of these suggestions for defining negation, show that if just EFQ and RAA are assumed as basic principles for negation, then CM, DNI, CON(i), CON(ii) can all be deduced.

5.4.4. What is called intuitionist logic differs from the classical two-valued logic primarily over its treatment of negation. At any rate, an intuitionist logic[3] for $\rightarrow$ and $\neg$ can be axiomatized by adding to our axiom-schemas (A1) and (A2) two further axiom-schemas for negation, corresponding to EFQ and RAA, i.e.

$$\vdash \varphi \rightarrow (\neg\varphi \rightarrow \psi)$$
$$\vdash (\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \neg\psi) \rightarrow \neg\varphi).$$

Alternatively, it can be axiomatized by defining $\neg$ in terms of $\perp$, and adding to (A1) and (A2) the single axiom-schema

$$\vdash \perp \rightarrow \psi.$$

Thus intuitionist logic contains all the theses of the previous exercise (i.e. CM, DNI, CON(i)–(ii), in addition to EFQ and RAA).

Show that it does not contain any of the other theses on the diagram (i.e. CM*, DNE, CON(iii)–(iv), TND). [Argue first that it is sufficient to show that it does not contain DNE. Then show that tables III and VI of Section 5.2 verify all intuitionist axioms (and Modus Ponens), but do not verify DNE.]

[3] See also n. 8 in the Appendix to this chapter.

5.6. **Axioms for the Quantifiers**

By being very generous over what to count as an axiom, it is *possible* to present a logic for the quantifiers which still contains no rule of inference other than the familiar rule Modus Ponens. (See the appendix to this chapter.) But it does complicate matters quite noticeably, since the Deduction Theorem is then much more difficult to establish. Consequently, the more usual way of extending axioms (A1)–(A3), so as to cover quantifiers as well as truth-functors, adds not only new axioms but a new rule of inference also. The simplest such rule to add is the rule of *generalization*, in this form:

GEN: If $\vdash \varphi$ then $\vdash \forall\xi\varphi(\xi/\alpha)$.

As well as this new rule, one adds also these two axiom-schemas:

(A4) $\vdash \forall\xi\varphi \rightarrow \varphi(\alpha/\xi)$.
(A5) $\vdash \forall\xi(\psi\rightarrow\varphi) \rightarrow (\psi\rightarrow\forall\xi\varphi)$, provided ξ is not free in ψ.

As we shall eventually see (but not until Chapter 7), this provides a complete basis for the logic of quantifiers.

There is room for some variation in the statement of the rule GEN and the axiom (A4). As I have just set down these theses, φ represents any formula, ξ any variable, and α any name-letter. This formulation evidently presumes that name-letters do occur in the language we are considering, and it also presumes that we are confining attention to closed formulae, so that no variables occur free in a finished formula. If open formulae are to be permitted too, then in (A4) α should be replaced by τ, standing in for any *term* (i.e. name-letter or variable). We may also write τ in place of α in the rule GEN, though in fact this will not make any difference to the theorems that can be proved, except when name-letters do *not* occur in the language. (For in that case the rule can only be applied where φ contains a free variable.) It should also be noted that GEN and (A4), as stated here, do allow for there to be vacuous quantifiers. For example, if φ lacks the letter α, but $\vdash \varphi$, then according to GEN we shall also have $\vdash \forall\xi\varphi$ for any variable ξ whatever. (Recall that if φ lacks α then $\varphi(\xi/\alpha)$ is φ.) If formulae with vacuous quantifiers are not wanted, then a restriction should be added to GEN to prevent this. But note also that if vacuous quantifiers are permitted, then it is easy to show, from the rules and axioms, that they add nothing. For if ξ is not free in φ, then from (A4) we have at once

$$\vdash \forall\xi\varphi\rightarrow\varphi.$$

Conversely, from GEN we have

$$\vdash \forall\xi(\varphi\rightarrow\varphi),$$

and hence by (A5) and MP

$$\vdash \varphi\rightarrow\forall\xi\varphi.$$

Thus φ and $\forall\xi\varphi$ are provably equivalent, if the quantifier is vacuous.

The new axioms and rules are sound, as I shall show shortly; that is, they are all correct under the intended interpretation. We can also choose unintended interpretations which make some correct and some incorrect, in ways which will show their mutual independence. For example, if we interpret the universal quantifier so that $\forall\xi\varphi$ is always counted as false, for every formula φ, then the two axioms are correct, but the rule GEN is clearly incorrect. This shows that GEN is independent of the rest of the system, i.e. that there are formulae which cannot be proved on the basis of (A1)–(A5), with MP, but can be proved if GEN is added to this basis. (The formula $\forall x(Fx \rightarrow Fx)$ is an example.) Equally, if we interpret the quantifier so that $\forall\xi\varphi$ is always counted as true, then GEN and (A5) remain correct, but (A4) does not, and this shows the independence of (A4). Finally, if we take a as our only name-letter, and interpret $\forall\xi\varphi$ to mean the same as '$\varphi(a/\xi)$ is a necessary truth', then GEN and (A4) remain correct, but (A5) does not, since there is now no way in which the proviso on (A5) can be used to prevent unwanted inferences. I should perhaps add that the earlier arguments to show that (A1)–(A3) were independent of one another can easily be carried over to show that each of (A1)–(A3) is independent of all the rest of the enlarged system containing (A4)–(A5) and GEN in addition. To see this, we may again take a as our only name-letter and interpret $\forall\xi\varphi$ to mean just $\varphi(a/\xi)$, so that the quantifiers are doing no work at all. Then GEN is a triviality, and the axioms each take the form $\vdash \varphi \rightarrow \varphi$, which is verified by all the tables that we considered.

We shall naturally want to extend the deduction theorem so that we are entitled to use proofs from assumptions with formulae involving quantifiers. Now there is no problem here over the addition of two new axioms, (A4) and (A5). The proof given earlier in Section 5.2 relies upon the fact that the system does contain the axioms (A1) and (A2), but it does not matter to that proof what other axioms there might be in the system. So we can add as many more axioms as we like without in any way disturbing the deduction theorem. But with rules of inference the position is different, for the earlier proof relies on the fact that the system does have the rule MP, *and* on the fact that it has no other rule. For it presumes that in a proof from assumptions every line must be either an assumption or an axiom or a consequence of previous lines *by the rule MP*. If there are other rules to be considered too, for example GEN, then there are further cases that need to be considered.

As a matter of fact, the rule GEN cannot itself be used within proofs from assumptions. It is instructive here to bring out the contrast between generalization on the one hand and on the other hand the original rule of detachment. These rules are

DET: If $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$ then $\vdash \psi$.
GEN: If $\vdash \varphi$ then $\vdash \forall\xi\varphi(\xi/\alpha)$.

Each of them is framed as a rule for use in proofs from *no* assumptions, which is how axiomatic proofs were first conceived. But the rule of detachment can immediately be liberalized into Modus Ponens, which is designed to be used in proofs from assumptions:

MP: If $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \rightarrow \psi$ then $\Gamma \vdash \psi$.

On the other hand, the rule of generalization certainly cannot be liberalized in a similar way, to

?: If $\Gamma \vdash \varphi$ then $\Gamma \vdash \forall\xi\varphi(\xi/\alpha)$.

To take a simple counter-example, the sequent

$Fa \vdash Fa$

is, of course, correct, but if we apply the suggested rule to it, then we obtain

$Fa \vdash \forall xFx$,

which is certainly not correct. (From the assumption that *a* is *F* it evidently does not follow that everything else is *F* as well.) The reason why GEN, as first stated, is a correct rule could be put like this: if you can prove some formula containing the name *a on no assumptions*, then in particular you have made no assumptions about *a*, so *a* could be anything. That is, we could in the same way prove the same point about anything else. So our formula must hold of everything whatever, and that is what the rule GEN says. But the important premiss to this reasoning is not that the formula is proved on no assumptions *at all*, but rather that it is proved on no assumptions *about a*; that is what allows us to add '*a* could be anything'. The right way, then, to liberalize the rule GEN, so that it can be used in proofs from assumptions, is this:

($\forall$I) If $\Gamma \vdash \varphi$, and if α does not occur in Γ, then $\Gamma \vdash \forall\xi\varphi(\xi/\alpha)$.

(The label '($\forall$I)' stands for 'universal quantifier introduction', cf. 3.6.G.) It will be seen that just as DET is a special case of MP, namely the case where there are no assumptions, so also GEN is the same special case of ($\forall$I). Consequently, a proof from no assumptions that uses the rule ($\forall$I) is at the same time a proper axiomatic proof using only the rule GEN.

We can now return to the deduction theorem. A proof from assumptions is now to be a finite sequence of formulae, each of which is either an assumption, or one of the axioms (A1)–(A5), or a consequence of previous formulae either by the rule MP or by the rule ($\forall$I). The deduction theorem states that if we have a proof from assumptions showing that $\Gamma,\varphi \vdash \psi$, then this can be transformed into another proof showing that $\Gamma \vdash \varphi \rightarrow \psi$. By repeated steps of this transformation, any proof from assumptions can therefore be transformed into a fully conditionalized proof, which is then an axiomatic proof as first defined. To prove the deduction theorem we must invoke axioms (A1) and (A2) to cover a case where the rule MP is applied, and axiom (A5) to cover a case where the rule ($\forall$I) is applied. The proof is just the same as before (p. 203), except for the extra case for the rule ($\forall$I), which is this:

> *Case (4)*: ψ is $\forall\xi\chi(\xi/\alpha)$, obtained from a previous line χ by the rule ($\forall$I). Note that, since ($\forall$I) was applicable, the name α does not occur either in Γ or in φ. By inductive hypothesis there is a proof showing that $\Gamma \vdash \varphi\rightarrow\chi$. Since α is not Γ, we may apply ($\forall$I) to this, to obtain a proof showing that $\Gamma \vdash \forall\xi(\varphi\rightarrow\chi)(\xi/\alpha)$. Since α is not in φ, this last formula is $\forall\xi(\varphi\rightarrow\chi(\xi/\alpha))$, where ξ is not free in φ. So we may add the axiom $\forall\xi(\varphi\rightarrow\chi(\xi/\alpha)) \rightarrow (\varphi\rightarrow\forall\xi\chi(\xi/\alpha))$, and apply MP to get $\varphi\rightarrow\forall\xi\chi(\xi/\alpha)$, i.e. $\varphi\rightarrow\psi$. The result is a proof showing that $\Gamma \vdash \varphi\rightarrow\psi$.

This establishes the deduction theorem.

I proceed at once to an illustration of its use. Given the rule ($\forall$I), all that was needed to establish the deduction theorem was an application of the axiom (A5). It is also true conversely that, given ($\forall$I) and the deduction theorem, we can now deduce (A5) (though we also need to call on (A4) in the proof).

(A5) $\vdash \forall\xi(\varphi\rightarrow\psi) \rightarrow (\varphi\rightarrow\forall\xi\psi)$, provided ξ is not free in φ

1. $\forall\xi(\varphi\rightarrow\psi)$	~~ASS~~(10)
2. $\forall\xi(\varphi\rightarrow\psi) \rightarrow (\varphi\rightarrow\psi)(\alpha/\xi)$	A4; choose α so that it is not in φ or ψ.
3. $(\varphi\rightarrow\psi)(\alpha/\xi)$	1,2,MP.
4. $\varphi\rightarrow\psi(\alpha/\xi)$	This is line (3), assuming ξ is not free in φ.
5. φ	~~ASS~~(9)
6. $\psi(\alpha/\xi)$	4,5,MP
7. $\forall\xi\psi(\alpha/\xi)(\xi/\alpha)$	6,$\forall$I; α is not in lines (1) or (5)
8. $\forall\xi\psi$	This is line (7), since α is not in ψ
9. $\varphi\rightarrow\forall\xi\psi$	5–8,D
10. $\forall\xi(\varphi\rightarrow\psi) \rightarrow (\varphi\rightarrow\forall\xi\psi)$	1–9,D

This proof-schema shows that, once we are given the deduction theorem, the basis consisting of the axiom (A4) with the rule ($\forall$I) is equivalent to our original basis consisting of (A4) and (A5) and the rule GEN. That is why I did not pause to prove the soundness of (A4), (A5), and GEN when they were first introduced, for the soundness of (A4) and ($\forall$I) has been proved already, on pp. 97–9.

Some features of this proof-schema deserve comment. Since it is a *schema*, and not an actual proof of a particular formula, it contains things which would not appear in an actual proof. The most obvious example is that what is written on line (3) is different from what is written on line (4), though the actual formula represented is exactly the same in each case. The same applies to lines (7) and (8). Since this kind of thing can be very distracting to one who is not already familiar with proofs of this sort, it is better, to begin with, to practise on actual proofs with actual formulae. This also eliminates such things as the instruction 'choose α so that it is not in φ or ψ' attached to line (2), for instead of putting in the instruction, we simply conform to it. For example, here is something more like a genuine proof of an instance of (A5):

$\forall x(P \rightarrow Fx) \rightarrow (P \rightarrow \forall xFx)$

1. $\forall x(P \rightarrow Fx)$	~~ASS~~(8)
2. $\forall x(P \rightarrow Fx) \rightarrow (P \rightarrow Fa)$	A4
3. $P \rightarrow Fa$	1,2,MP
4. P	~~ASS~~(7)
5. Fa	3,4,MP
6. $\forall xFx$	5,$\forall$I; a is not in lines (1) or (4)
7. $P \rightarrow \forall xFx$	4–6,D
8. $\forall x(P \rightarrow Fx) \rightarrow (P \rightarrow \forall xFx)$	1–7,D

Here lines (1)–(6) do constitute a genuine proof from assumptions, though, of course, lines (7) and (8) merely indicate that this can be transformed into a proof from no assumptions; they do not carry out that transformation. Nevertheless, the proof now looks more like the kind of examples that we have had earlier in this chapter, except that the justification for line (6) is untypically long. This is because the rule ($\forall$I) is a *conditional* rule, for it allows you to introduce a universal quantifier *on the condition that* the name that is to be generalized does not occur in the assumptions to the proof. So whenever the rule is applied one must look back at all the previous lines labelled 'Assumption', and check that this condition is indeed satisfied. (Assumptions which have *already* been discharged can, of course, be ignored.) When one has grown used to checking in this way, then no doubt one can save time by not bothering to write in explicitly that the condition is satisfied, but still the check must, of course, be done.

I close this section with two more examples of proofs involving quantifiers, to establish a thesis first used earlier on p. 120, namely

$$\neg\forall x\neg(Fx \to Gx) \dashv\vdash \forall xFx \to \neg\forall x\neg Gx.$$

(But here I have to write '$\neg\forall x\neg$' in place of the shorter '$\exists x$' used earlier.)

(a) $\neg\forall x\neg(Fx\to Gx) \vdash \forall xFx \to \neg\forall x\neg Gx$

1.	$\neg\forall x\neg(Fx\to Gx)$	ASS
2.	$\forall xFx$	~~ASS~~(11)
3.	$\forall x\neg Gx$	~~ASS~~(8)
4.	Fa	2,A4,MP
5.	$\neg Ga$	3,A4,MP
6.	$\neg(Fa\to Ga)$	4,5,Exercise 5.5.1(a)
7.	$\forall x\neg(Fx\to Gx)$	6,$\forall$I; a is not in lines (1)–(3)
8.	$\forall x\neg Gx \to \forall x\neg(Fx\to Gx)$	3–7,D
9.	$\neg\forall x\neg(Fx\to Gx) \to \neg\forall x\neg Gx$	8,CON(i)
10.	$\neg\forall x\neg Gx$	1,9,MP
11.	$\forall xFx \to \neg\forall x\neg Gx$	2–10,D

(b) $\forall xFx \to \neg\forall x\neg Gx \vdash \neg\forall x\neg(Fx\to Gx)$

1.	$\forall xFx \to \neg\forall x\neg Gx$	ASS
2.	$\forall x\neg(Fx\to Gx)$	~~ASS~~(8)
3.	$\neg(Fa\to Ga)$	2,A4,MP
4.	Fa	3,Exercise 5.5.1(a)
5.	$\neg Ga$	3,Exercise 5.5.1(a)
6.	$\forall xFx$	4,$\forall$I; a is not in lines (1)–(2)
7.	$\forall x\neg Gx$	5,$\forall$I; a is not in lines (1)–(2)
8.	$\forall x\neg(Fx\to Gx) \to \forall x\neg Gx$	2$\to$7,D
9.	$\neg\forall x\neg Gx \to \neg\forall x\neg(Fx\to Gx)$	8, CON(i)
10.	$\neg\forall x\neg Gx$	1,6,MP
11.	$\neg\forall x\neg(Fx\to Gx)$	9,10,MP.

These proofs should be studied before turning to the exercises that follow.

EXERCISES

5.6.1. Provide proofs of the following. (You may use in the proofs any sequent that has been proved earlier in this chapter—or indeed, in view of the completeness of the axioms for $\to$ and $\neg$, any sequent that can be established by truth-tables.)

(a) $\forall x(Fx\to Gx) \vdash \forall xFx \to \forall xGx$.
(b) $\forall x(Fx\to Gx) \vdash \neg\forall x\neg Fx \to \neg\forall x\neg Gx$.
(c) $\forall x(P\to Fx) \dashv\vdash P \to \forall xFx$.

(*d*) $\forall x(Fx \rightarrow P) \dashv\vdash \neg\forall x\neg Fx \rightarrow P$.
(*e*) $\neg\forall x\neg\forall yFxy \vdash \forall y\neg\forall x\neg Fxy$.

[If (*e*) proves difficult, then try it again after reading the next section.]

5.6.2. (continuing 5.5.2). Assume (as is the case) that our axiom-schemas (A1)–(A5), together with MP and GEN, form a complete basis for the logic of quantifiers. Show that it is nevertheless *not true* that if we add to them any further axiom-schema of the language, not already provable from them, then the result is an inconsistent system. [Hint: consider the schema $\forall\xi\varphi \vee \forall\xi\neg\varphi$.]

5.7. Definitions of Other Logical Symbols

As we have noted (p. 46), the zero-place truth-functors $\top$ and $\bot$ cannot strictly be defined in terms of any other truth-functors. But usually one does not speak so strictly, and accepts definitions such as

$\top$ for $P \rightarrow P$.
$\bot$ for $\neg(P \rightarrow P)$.

It is easily seen that these definitions give rise to the rules characteristic of $\top$ and $\bot$, namely

$\varphi \vdash \top$.
$\bot \vdash \varphi$.

It may also be noted that if $\bot$ is available, then this gives us a simple way of defining sequents with no formula on the right, for we may abbreviate

$\Gamma \vdash$ for $\Gamma \vdash \bot$.

Putting these definitions together, we obtain the 'structural' rule of Thinning on the right

If $\Gamma \vdash$ then $\Gamma \vdash \varphi$.

We can also, if we wish, restrict EFQ to this

EFQ′: $\varphi, \neg\varphi \vdash \bot$.

And a rather nice form of *reductio ad absurdum* becomes available, namely

RAA′: If $\Gamma, \varphi \vdash \bot$ then $\Gamma \vdash \neg\varphi$.

This will be used in what follows. Notice that, like applications of the deduction theorem, this is a rule that *discharges* an assumption. For if φ was an assumption, and from it we obtain a contradiction $\bot$, then we can infer $\neg\varphi$ and drop φ from the assumptions. (The justification of this form of RAA is left as an exercise.)

The functors $\top$ and $\bot$ may be regarded as something of a luxury; at any rate they have little work to do in the usual applications of logical theory to test actual arguments. Here the functors $\wedge$ and $\vee$ are very much in demand, and we may, of course, define them in terms of $\rightarrow$ and $\neg$ by putting

$\varphi\wedge\psi$ for $\neg(\varphi\rightarrow\neg\psi)$.
$\varphi\vee\psi$ for $\neg\varphi\rightarrow\psi$.

But we shall not be able to do much with these functors until we have proved for them some suitable rules. In the case of $\wedge$, the rules are these:

$$(\wedge I)\ \varphi,\psi\vdash\varphi\wedge\psi \qquad (\wedge E)\ \varphi\wedge\psi\vdash\varphi,\quad \varphi\wedge\psi\vdash\psi.$$

(The labels '($\wedge$I)' and '($\wedge$E)' are short for '$\wedge$-introduction' and '$\wedge$-elimination'.) These rules (once we have proved them) are to be used in proofs from assumptions in just the way that the various rules for negation were used in Section 5.3. Proofs are as follows. (These proofs use RAA and EFQ in the new forms, and in addition DNE and the principle $\neg\varphi\vdash\varphi\rightarrow\psi$, for which I cite EFQ plus one step of the deduction theorem.)

($\wedge$I) $\varphi,\psi\vdash\varphi\wedge\psi$

1. φ	ASS
2. ψ	ASS
3. $\varphi\rightarrow\neg\psi$	~~ASS~~(6)
4. $\neg\psi$	1,3,MP
5. $\bot$	2,4,EFQ′
6. $\neg(\varphi\rightarrow\neg\psi)$	3–5,RAA′
7. $\varphi\wedge\psi$	6,Def$\wedge$

($\wedge$E) $\varphi\wedge\psi\vdash\varphi$

1. $\varphi\wedge\psi$	ASS
2. $\neg(\varphi\rightarrow\neg\psi)$	1,Def$\wedge$
3. $\neg\varphi$	~~ASS~~(6)
4. $\varphi\rightarrow\neg\psi$	3,EFQ,D
5. $\bot$	2,4,EFQ′
6. $\neg\neg\varphi$	3–5, RAA′
7. φ	6,DNE

($\wedge$E) $\varphi \wedge \psi \vdash \psi$

1. $\varphi \wedge \psi$	ASS
2. $\neg(\varphi \rightarrow \neg\psi)$	1,Def$\wedge$
3. $\neg\psi$	~~ASS~~(6)
4. $\varphi \rightarrow \neg\psi$	3,D
5. $\perp$	2,4,EFQ′
6. $\neg\neg\psi$	3–5,RAA′
7. ψ	6,DNE

In the case of $\vee$, it is easy to state suitable introduction rules, namely

($\vee$I) $\varphi \vdash \varphi \vee \psi$, $\psi \vdash \varphi \vee \psi$.

It is more difficult to frame a suitable 'elimination' rule to accompany these. For the moment, I shall put it in this way:

($\vee$E) $\varphi \rightarrow \chi, \psi \rightarrow \chi \vdash \varphi \vee \psi \rightarrow \chi$.

(But a different version, which more clearly justifies the title '($\vee$E)', will be introduced shortly.) Here are proofs:

($\vee$I) $\varphi \vdash \varphi \vee \psi$

1. φ	ASS
2. $\neg\varphi \rightarrow \psi$	1,EFQ,D
3. $\varphi \vee \psi$	2,Def$\vee$

($\vee$I) $\psi \vdash \varphi \vee \psi$

1. ψ	ASS
2. $\neg\varphi \rightarrow \psi$	1,D
3. $\varphi \vee \psi$	2,Def$\vee$

($\vee$E) $\varphi \rightarrow \chi, \psi \rightarrow \chi \vdash \varphi \vee \psi \rightarrow \chi$

1. $\varphi \rightarrow \chi$	ASS
2. $\psi \rightarrow \chi$	ASS
3. $\varphi \vee \psi$	~~ASS~~(10)
4. $\neg\varphi \rightarrow \psi$	3,Def$\vee$
5. $\neg\varphi$	~~ASS~~(8)
6. ψ	4,5,MP
7. χ	2,6,MP
8. $\neg\varphi \rightarrow \chi$	5–7,D
9. χ	1,8,TND
10. $\varphi \vee \psi \rightarrow \chi$	3–9,D

This has shown that our axioms (A1)–(A3), together with suitable definitions of $\wedge$ and $\vee$, will allow us to deduce suitable rules for using those functors in proofs. The reverse is also true. That is, if we add to the original axioms the introduction and elimination rules for $\wedge$ and $\vee$, then we can deduce the defining equivalences. (The proof of this is left as an exercise.)

Finally in this chapter let us consider the definition of $\exists$, namely

$$\exists\xi \quad \text{for} \quad \neg\forall\xi\neg.$$

We shall show that this definition yields these rules:

($\exists$I) $\varphi(\alpha/\xi) \vdash \exists\xi\varphi$

($\exists$E) If $\Gamma,\varphi \vdash \psi$, and if α is not in Γ or in ψ, then $\Gamma,\exists\xi\varphi(\xi/\alpha) \vdash \psi$

Here are proofs:

($\exists$I) $\varphi(\alpha/\xi) \vdash \exists\xi\varphi$

1. $\varphi(\alpha/\xi)$	ASS
2. $\forall\xi\neg\varphi$	~~ASS~~(5)
3. $\neg\varphi(\alpha/\xi)$	2,A4,MP
4. $\bot$	1,3,EFQ′
5. $\neg\forall\xi\neg\varphi$	2–4,RAA′
6. $\exists\xi\varphi$	4,Def$\exists$

($\exists$E) If $\Gamma,\varphi \vdash \psi$ then $\Gamma,\exists\xi\varphi(\xi/\alpha) \vdash \psi$, provided α is not in Γ or in ψ

1. Γ	ASS
2. $\exists\xi\varphi(\xi/\alpha)$	ASS
3. $\neg\forall\xi\neg\varphi(\xi/\alpha)$	2,Def$\exists$
4. $\neg\psi$	~~ASS~~(11)
5. φ	~~ASS~~(8)
6. ψ	1,5,Hypothesis
7. $\bot$	4,6
8. $\neg\varphi$	5–7,RAA′
9. $\forall\xi\neg\varphi(\xi/\alpha)$	8,$\forall$I; α is not in lines (1),(2),(4)
10. $\bot$	3,9,EFQ′
11. $\neg\neg\psi$	4–10, RAA′
12. ψ	11,DNE

EXERCISES

5.7.1. Show that the deductions of this section can be reversed, i.e. that if we assume as premisses the rules ($\wedge$I),($\wedge$E),($\vee$I),($\vee$E),($\exists$I),($\exists$E), then we can deduce from them equivalences corresponding to the definitions of $\wedge$,$\vee$,$\exists$, namely

$\varphi\wedge\psi \quad \dashv\vdash \quad \neg(\varphi\rightarrow\neg\psi).$
$\varphi\vee\psi \quad \dashv\vdash \quad \neg\varphi\rightarrow\psi.$
$\exists\xi\varphi \quad \dashv\vdash \quad \neg\forall\xi\neg\varphi.$

5.7.2. Consider an axiomatic system which has $\rightarrow$ as its only truth-functor, the rule of detachment as its only rule, and the following three axiom-schemas:

1. $\varphi\rightarrow(\psi\rightarrow\varphi)$.
2. $(\varphi\rightarrow(\psi\rightarrow\chi)) \rightarrow ((\varphi\rightarrow\psi)\rightarrow(\varphi\rightarrow\chi))$.
3. $((\varphi\rightarrow\psi)\rightarrow\varphi) \rightarrow \varphi$.

You may assume that proof from assumptions is defined, and the deduction theorem proved (using axioms (1) and (2)). Introduce the functor $\vee$ by defining

$\varphi\vee\psi$ for $(\varphi\rightarrow\psi)\rightarrow\psi$

and prove the following:

4. $\varphi\vdash\varphi\vee\psi$
5. $\varphi\vee\psi\vdash\psi\vee\varphi$ [See Exercise 5.3.3.]
6. $\varphi\rightarrow\psi,\varphi\vee\chi\vdash\psi\vee\chi$
7. $\varphi\rightarrow\chi,(\varphi\rightarrow\psi)\rightarrow\chi\vdash\chi$ [Use 3.]
8. $\varphi\rightarrow\chi,\psi\rightarrow\chi\vdash(\varphi\vee\psi)\rightarrow\chi$ [Use 7.]
9. $(\varphi\vee\psi)\vee\chi\vdash\varphi\vee(\psi\vee\chi)$ [Use 4,5,8.]
10. If $\Gamma,\varphi\vdash\chi$ and $\Gamma\vdash\varphi\vee\chi$ then $\Gamma\vdash\chi$
11. If $\Gamma,\varphi\vdash\chi$ then $\Gamma\vdash\chi\vee(\varphi\rightarrow\psi)$
12. If $\Gamma\vdash\chi\vee\psi$ then $\Gamma\vdash\chi\vee(\varphi\rightarrow\psi)$ [Use 6.]
13. If $\Gamma,\psi\vdash\chi$ and $\Gamma\vdash\varphi\vee\chi$ then $\Gamma,\varphi\rightarrow\psi\vdash\chi$ [Use 6,10.]

This is the beginning of a proof to show that axioms (1)–(3) form a complete basis for all valid sequents whose only truth-functor is $\rightarrow$. To assist comprehension, let us write '$\Gamma\vdash\Delta$' to mean 'from the set of formulae in Γ there is a proof of the disjunction of all the formulae in Δ'. Then (5) and (9) tell us that the order and grouping of the disjunction represented by Δ may be ignored, and (10)–(13) may be rewritten thus:

10′ If $\Gamma,\varphi\vdash\Delta$ and $\Gamma\vdash\varphi,\Delta$ then $\Gamma\vdash\Delta$
11′ If $\Gamma,\varphi\vdash\Delta$ then $\Gamma\vdash\varphi\rightarrow\psi,\Delta$
12′ If $\Gamma\vdash\psi,\Delta$ then $\Gamma\vdash\varphi\rightarrow\psi,\Delta$
13′ If $\Gamma,\psi\vdash\Delta$ and $\Gamma\vdash\varphi,\Delta$ then $\Gamma,\varphi\rightarrow\psi\vdash\Delta$.

(As usual we write φ,Δ as short for $\{\varphi\}\cup\Delta$.) To see how the proof continues from here, consult Exercise 7.4.5.

5.7.3. Add to the system of the previous exercise the truth-functor $\bot$, and a single axiom-schema for it, to give the following set of axiom-schemas:

1. $\varphi\rightarrow(\psi\rightarrow\varphi)$.
2. $(\varphi\rightarrow(\psi\rightarrow\chi)) \rightarrow ((\varphi\rightarrow\psi)\rightarrow(\varphi\rightarrow\chi))$.
3. $((\varphi\rightarrow\psi)\rightarrow\varphi) \rightarrow \varphi$.
4. $\bot\rightarrow\varphi$.

Define negation by putting

$\neg\varphi$ for $\varphi\rightarrow\bot$.

Show that this set of axioms is a complete basis for the logic of truth-functors. [Method: show that from (3) and (4) and the definition one can deduce both EFQ and CM*.]

5.8. Appendix: Some Alternative Axiomatizations

The first axiomatization of logic was in Frege's *Begriffsschrift* of 1879. His axioms for the truth-functors were[5]

1. $\varphi\rightarrow(\psi\rightarrow\varphi)$.
2. $(\varphi\rightarrow(\psi\rightarrow\chi)) \rightarrow ((\varphi\rightarrow\psi)\rightarrow(\varphi\rightarrow\chi))$.
3. $(\varphi\rightarrow(\psi\rightarrow\chi)) \rightarrow (\psi\rightarrow(\varphi\rightarrow\chi))$.
4. $(\varphi\rightarrow\psi) \rightarrow (\neg\psi\rightarrow\neg\varphi)$.
5. $\neg\neg\varphi\rightarrow\varphi$.
6. $\varphi\rightarrow\neg\neg\varphi$.

It was later shown that his third axiom was superfluous, since it can be derived from the first two (Łukasiewicz 1936). Another early axiomatization, which became widely known, was that of Russell and Whitehead's *Principia Mathematica* (vol. i, 1910), which takes $\neg$ and $\vee$ as its basic vocabulary, but at once introduces $\rightarrow$ by the usual definition. The axioms are

1. $\varphi\vee\varphi \rightarrow \varphi$.
2. $\psi \rightarrow \varphi\vee\psi$.
3. $\varphi\vee\psi \rightarrow \psi\vee\varphi$.
4. $\varphi\vee(\psi\vee\chi) \rightarrow (\varphi\vee\psi)\vee\chi$.
5. $(\psi\rightarrow\chi) \rightarrow (\varphi\vee\psi\rightarrow\varphi\vee\chi)$.

It was later shown that the fourth axiom was superfluous (Bernays 1926), and in fact with minor changes elsewhere both (3) and (4) can be rendered superfluous (Rosser 1953). But in any case this is an unsatisfying set of axioms, for their purport is reasonably clear only when (as here) one uses *both* $\rightarrow$ *and* $\vee$ in the formulation. But officially, the primitive notation is just $\neg$ and $\vee$, and when $\rightarrow$ is replaced by this primitive notation the axioms seem very arbitrary indeed.

In almost all cases,[6] axioms for the truth-functors are designed to be used with the rule of detachment as the sole rule of inference, and that rule is naturally

[5] Strictly speaking, Frege did not use axiom-schemas, as here, but single axioms and a rule of substitution. (The idea of an axiom-schema was due to J. von Neumann (1927).) I have consistently ignored this distinction in this appendix.

[6] An exception is noted below, where axioms and rules are designed for the stroke functor.

formulated with $\rightarrow$. One therefore expects to find $\rightarrow$ in the primitive vocabulary, and playing an important role in the axioms. If this is granted, then there are broadly speaking two approaches to choose between. One may aim for economy in the axioms, by restricting the language. In that case one will naturally choose a language with just $\rightarrow$ and $\neg$, or just $\rightarrow$ and $\bot$. Or one may say that it is much more convenient in practice to have a richer language, and a correspondingly rich set of axioms. I pursue each of these suggestions in turn.

The system in $\rightarrow$ and $\neg$ that we have used in this chapter is easily seen to be a descendant of Frege's original system. It retains his first two axioms for $\rightarrow$ alone, omitting his third as superfluous, and adds to these one further axiom for negation. We have already explored (in Section 5.4) various other possibilities for the negation axioms, and do not need to add anything more here. A variation is to add axioms for $\bot$ rather than for $\neg$, and here we find that a single axiom which will do by itself is

$$((\varphi\rightarrow\bot)\rightarrow\bot) \rightarrow \varphi.$$

This yields a nicely economical system (which is used by Church 1956).

There is nothing in this general approach that forces us to retain Frege's first two axioms. It is true that they are very convenient, since they are just what we need to prove the deduction theorem (which was not known to Frege),[7] and this is a great help in finding proofs. But (*a*) this theorem can of course be postponed, if other axioms prove more attractive, and (*b*) there is the objection that these two axioms for $\rightarrow$ are not strong enough as they stand, since they do not suffice by themselves for the proof of all correct sequents concerning $\rightarrow$ on its own (cf. Exercise 5.3.3). In response to (*a*) there are various sets of axioms known to be equivalent to Frege's first two, for example this set of three:

1. $\varphi\rightarrow(\psi\rightarrow\varphi)$.
2. $(\varphi\rightarrow(\varphi\rightarrow\psi)) \rightarrow (\varphi\rightarrow\psi)$.
3. $(\varphi\rightarrow\psi) \rightarrow ((\psi\rightarrow\chi) \rightarrow (\varphi\rightarrow\chi))$.

(See Exercise 5.8.2.) But I am not aware of any set that seems more simple or more attractive than Frege's own pair. In response to (*b*) the most straightforward suggestion is just to *add* to Frege's pair a further axiom, for example Peirce's law:

$$((\varphi\rightarrow\psi)\rightarrow\varphi) \rightarrow \varphi.$$

This provides a set of three axioms for $\rightarrow$ which do suffice for the deduction of all correct sequents whose only functor is $\rightarrow$. (See Exercises 5.7.2. and 7.4.5.) An alternative with the same effect is to allow Peirce's law to replace the second axiom in the trio just cited, to yield

1. $\varphi\rightarrow(\psi\rightarrow\varphi)$
2. $((\varphi\rightarrow\psi)\rightarrow\varphi) \rightarrow \varphi$
3. $(\varphi\rightarrow\psi) \rightarrow ((\psi\rightarrow\chi) \rightarrow (\varphi\rightarrow\chi))$

[7] It is due to Herbrand (1930).

(Łukasiewicz and Tarski 1930). As it turns out, we can provide a *single* axiom which is adequate on its own for all correct sequents whose only functor is $\rightarrow$, namely

$$((\varphi\rightarrow\psi_1)\rightarrow\chi) \rightarrow ((\chi\rightarrow\varphi)\rightarrow(\psi_2\rightarrow\varphi))$$

(Łukasiewicz 1948). But this does not exactly strike one as a perspicuous axiom, and it is not at all easy to work with. Finally I add here that if we do adopt axioms for $\rightarrow$ which suffice for the deduction of all valid sequents in $\rightarrow$, then we need add only one simple axiom for $\perp$ to obtain a complete system, namely

$$\perp\rightarrow\varphi.$$

The axiom system adopted in this chapter can obtain many results for $\rightarrow$ without calling upon its axiom for negation, but not all, as we have seen. One could shift this balance in the other direction by strengthening the negation axioms, relying even more upon them for results which concern $\rightarrow$ alone, and consequently weakening the axioms for $\rightarrow$. An interesting system which does just this is based on the three axioms

$$(\varphi\rightarrow\psi) \rightarrow ((\psi\rightarrow\chi) \rightarrow (\varphi\rightarrow\chi))$$
$$(\neg\varphi\rightarrow\varphi)\rightarrow\varphi$$
$$\varphi\rightarrow(\neg\varphi\rightarrow\psi)$$

(Łukasiewicz 1936). Pursuing this direction further, and making no attempt to distinguish between axioms for $\rightarrow$ and axioms for $\neg$, we can in fact make do with a single axiom, namely

$$((((\varphi_1\rightarrow\varphi_2)\rightarrow(\neg\psi_2\rightarrow\neg\psi_1))\rightarrow\psi_2)\rightarrow\chi) \rightarrow ((\chi\rightarrow\varphi_1)\rightarrow(\psi_1\rightarrow\varphi_1))$$

(Meredith 1953). Like all single axioms, it is neither perspicuous nor easy to work with.

(This is perhaps the place to mention that the first single axiom for the logic of truth-functors was found as long ago as 1917, by J. Nicod. His axiom is formulated for the Sheffer stroke (p. 58), and is

$$[\varphi_1\uparrow(\varphi_2\uparrow\varphi_3)]\uparrow(\{\chi\uparrow(\chi\uparrow\chi)]\uparrow\{(\psi\uparrow\varphi_2)\uparrow[(\varphi_1\uparrow\psi)\uparrow(\varphi_1\uparrow\psi)]\}).$$

This axiom is designed to be used, not with the usual rule of detachment for $\rightarrow$, but with a special rule for the Sheffer stroke, namely

$$\text{If } \vdash \varphi \text{ and } \vdash \varphi\uparrow(\psi\uparrow\chi) \text{ then } \vdash \chi.$$

(This is a generalization of the usual rule, for Detachment itself corresponds to the special case of this where ψ and χ are identified.) Since Nicod, some other versions of his single axiom have been found, which are equally long but perhaps do have a marginally better claim to elegance. But in any case single axioms seem to me to be a mere curiosity.)

To sum up on $\rightarrow$ and $\neg$, or $\rightarrow$ and $\perp$, it will be seen that although there is plenty of choice on which axioms to adopt, there is no choice which stands out as the most simple and straightforward, or the most elegant, or the one that most reveals our understanding of the functors involved. There is nothing here to rival the simple and very obvious rules of the semantic tableaux:

$$\begin{array}{ccc} & |\varphi\rightarrow\psi| = T & \\ & | & \\ |\varphi| = F & & |\psi| = T \end{array} \qquad \begin{array}{c} |\varphi\rightarrow\psi| = F \\ | \\ |\varphi| = T \\ |\psi| = F \end{array}$$

$$\begin{array}{c} |\neg\varphi| = T \\ | \\ |\varphi| = F \end{array} \qquad\qquad \begin{array}{c} |\neg\varphi| = F \\ | \\ |\varphi| = T \end{array}$$

This awkward situation will gradually be improved during the next two chapters. I now turn to a brief account of the alternative approach to axiomatization.

We may wish to consider from the start a richer language with more truth-functors, and a correspondingly richer set of axioms for those functors. For example, the following rather nice set is used by Kleene (1952):

1. $\varphi\rightarrow(\psi\rightarrow\varphi)$.
2. $(\varphi\rightarrow(\psi\rightarrow\chi)) \rightarrow ((\varphi\rightarrow\psi)\rightarrow(\varphi\rightarrow\chi))$.
3. $\varphi\wedge\psi\rightarrow\varphi$.
4. $\varphi\wedge\psi\rightarrow\psi$.
5. $\varphi\rightarrow(\psi\rightarrow\varphi\wedge\psi)$.
6. $\varphi\rightarrow\varphi\vee\psi$.
7. $\psi\rightarrow\varphi\vee\psi$.
8. $(\varphi\rightarrow\chi) \rightarrow ((\psi\rightarrow\chi) \rightarrow (\varphi\vee\psi\rightarrow\chi))$.
9. $(\varphi\rightarrow\psi) \rightarrow ((\varphi\rightarrow\neg\psi) \rightarrow \neg\varphi)$.
10. $\neg\neg\varphi\rightarrow\varphi$.

Here axioms (1) and (2) are our standard (but incomplete) axioms for $\rightarrow$, and axioms (9) and (10) are a standard pair of axioms for $\neg$. But (3)–(5) add new axioms for $\wedge$, and (6)–(7) add new axioms for $\vee$. (We could restore the symmetry between these two new sets of axioms by rewriting (5) as

5′. $(\varphi\rightarrow\psi) \rightarrow ((\varphi\rightarrow\chi) \rightarrow (\varphi\rightarrow\psi\wedge\chi))$.

This would make no difference to the resulting theorems.) Axioms in this style were first introduced by Hilbert and Bernays (1934). (Their version adopts the modification just suggested, and it also has different but equivalent versions of axioms (1)–(2) and (9)–(10).) I observe here merely that the proposed axioms for $\wedge$ and $\vee$ lead very directly into the methods pursued in the next chapter, so I reserve further comment until then.[8]

Turning to the quantifiers, almost all systems adopt our axiom (A4):

[8] The full version of intuitionist logic, without quantifiers, is given by axioms (1)–(9), with EFQ in place of (10). The version used earlier in Exercises 5.4.3 and 5.4.4 is a truncated version, since it does not include $\wedge$ and $\vee$, and in intuitionist logic these cannot be defined in terms of $\rightarrow$ and $\neg$.

$\vdash \forall\xi\varphi \rightarrow \varphi(\alpha/\xi)$.

Then there is a choice, either to add also our axiom (A5) and the simple rule GEN, or to add a more complex rule from which both of these are deducible, namely

If $\vdash \varphi\rightarrow\psi$ then $\vdash \varphi \rightarrow \forall\xi\varphi(\xi/\alpha)$, provided that α is not in φ.

(Frege himself adopted both this and GEN, but in his system GEN is superfluous.) The existential quantifier can then be defined in terms of the universal one, or it can be introduced by a dual pair of one axiom and one rule:

$\vdash \varphi(\alpha/\xi) \rightarrow \exists\xi\varphi$.

If $\vdash \varphi\rightarrow\psi$ then $\vdash \exists\xi\varphi(\xi/\alpha) \rightarrow \psi$, provided that α is not in ψ.

This technique is again closely related to that which will be pursued in the next chapter.

EXERCISES

5.8.1. Consider Church's system for $\rightarrow$ and $\perp$, which has these three axioms:

1. $\varphi\rightarrow(\psi\rightarrow\varphi)$.
2. $(\varphi\rightarrow(\psi\rightarrow\chi)) \rightarrow ((\varphi\rightarrow\psi)\rightarrow(\varphi\rightarrow\chi))$.
3. $((\varphi\rightarrow\perp)\rightarrow\perp) \rightarrow \varphi$.

Show how to modify the completeness proof of Section 5.5 to prove that in this system every valid formula whose only truth-functors are $\rightarrow$ and $\perp$ is a theorem.

5.8.2. Consider the system given by these three axioms:

1. $\varphi\rightarrow(\psi\rightarrow\varphi)$.
2. $(\varphi\rightarrow(\varphi\rightarrow\psi)) \rightarrow (\varphi\rightarrow\psi)$.
3. $(\varphi\rightarrow\psi) \rightarrow ((\psi\rightarrow\chi) \rightarrow (\varphi\rightarrow\chi))$.

In this system, prove the following theorems:

4. $\varphi\rightarrow((\varphi\rightarrow\psi)\rightarrow\psi)$.
5. $(\varphi\rightarrow(\psi\rightarrow\chi)) \rightarrow (\psi\rightarrow(\varphi\rightarrow\chi))$.
6. $(\psi\rightarrow\chi) \rightarrow ((\varphi\rightarrow\psi) \rightarrow (\varphi\rightarrow\chi))$.
7. $(\varphi\rightarrow(\psi\rightarrow\chi)) \rightarrow ((\varphi\rightarrow\psi)\rightarrow(\varphi\rightarrow\chi))$.

Deduce that this system is equivalent to that given by our axioms (A1) and (A2). [Hints: we cannot assume that the deduction theorem applies to this system until we have proved (7), so this exercise calls for genuinely axiomatic proofs to be constructed. Since this is far from easy, I give some help. Here is a proof of (4) in a much abbreviated form:

$\varphi\rightarrow((\varphi\rightarrow\psi)\rightarrow\varphi)$	from (1)
$\rightarrow((\varphi\rightarrow\psi) \rightarrow ((\varphi\rightarrow\psi)\rightarrow\psi))$	from (3)
$\rightarrow((\varphi\rightarrow\psi)\rightarrow\psi)$	from (2)

Show how to reconstruct the full proof from this sketch. To construct a similar sketch for (5) you will need to use

$[\varphi\rightarrow(\psi\rightarrow\chi)] \rightarrow [((\psi\rightarrow\chi)\rightarrow\chi) \rightarrow (\varphi\rightarrow\chi)]$	from (3)
$[((\psi\rightarrow\chi)\rightarrow\chi) \rightarrow (\varphi\rightarrow\chi)] \rightarrow [\psi\rightarrow(\varphi\rightarrow\chi)]$	from (4) and (3)

Given (5), it is easy to prove (6) from (3). The proof of (7) begins by stating (5) and then using this instance of (6):

$$[\psi\rightarrow(\varphi\rightarrow\chi)] \rightarrow [(\varphi\rightarrow\psi) \rightarrow (\varphi\rightarrow(\varphi\rightarrow\chi))].$$

5.8.3. Consider the system given by these three axioms:

1. $\varphi \rightarrow (\psi \rightarrow \varphi)$.
2. $((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi$.
3. $(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$.

In this system, prove the theorem

4. $(\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)$.

Deduce, using the previous exercise, that the deduction theorem holds for this system. [Hint: you will need this instance of axiom (2)]:

$$[((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow (\varphi \rightarrow \psi)] \rightarrow (\varphi \rightarrow \psi).$$

5.8.4. Consider the system got by adding, to any basis that is adequate for the truth-functors, these two further rules of inference:

1. If $\vdash \varphi \rightarrow \psi$ then $\vdash \forall \xi \varphi(\xi/\alpha) \rightarrow \psi$, provided that ξ is not free in φ.
2. If $\vdash \varphi \rightarrow \psi$ then $\vdash \varphi \rightarrow \forall \xi \psi(\xi/\alpha)$, provided that α does not occur in φ.

Show that this system is equivalent to the system of Section 5.6, which instead adds the two axioms (A4) and (A5) and the rule GEN.