7.4. **Gentzen Sequents; Semantic Tableaux Again**

We could pursue straightaway the comparison already hinted at between the sequent calculus corresponding to natural deduction and that corresponding to the tableau system. But at the moment negation is still playing a very special role in the tableau rules, and this is distracting. So I first adopt a new sequent calculus for the tableau system, which involves the use of a new kind of sequent altogether.

It is not an unreasonable suggestion that the unwanted occurrences of negation in many of the tableau rules can be removed if we recall that in the tableau system we abbreviate

$$\Gamma \Rightarrow \varphi \quad \text{for} \quad \Gamma, \neg\varphi \Rightarrow.$$

Applying this transformation to the rule for basic sequents, and to all the $\neg$-rules, they become

$$(\text{BS})\,\frac{}{\Gamma,\varphi \Rightarrow \varphi}$$

$$(\neg\wedge)\,\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi\wedge\psi}$$

$(\neg\vee)$?

$$(\neg\rightarrow)\,\frac{\Gamma,\varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi\rightarrow\psi}$$

$$(\neg\neg)\,\frac{\Gamma,\varphi \Rightarrow}{\Gamma \Rightarrow \neg\varphi}$$

$$(\neg\forall)\ \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \forall\xi\varphi(\xi/\alpha)}$$
provided α is not in Γ

$$(\neg\exists)\ \frac{\Gamma \Rightarrow \varphi(\alpha/\xi)}{\Gamma \Rightarrow \exists\xi\varphi}$$

In every case, this turns a complex and unfamiliar negation rule into a simpler and more familiar rule, for introducing on the right, with no superfluous intrusion of negations. But there is one case, namely ($\neg\vee$), to which the transformation cannot be applied, since it would lead to a sequent with *two* formulae to the right of $\Rightarrow$. It is true that one might try to avoid this by reformulating ($\neg\vee$) as in Exercise 7.3.4, but we noted then that that introduced further complications of its own. So what we shall do now is to enlarge the notion of a sequent, so that it may have any (finite) number of formulae on the left *and* any (finite) number on the right. Such sequents are called Gentzen sequents, for they were introduced by Gerhard Gentzen (1934).

The idea, then, is that $\Gamma \Rightarrow \Delta$ will be a sequent, where both Γ and Δ may be lists of several formulae (or of none). The intended interpretation is that such a sequent will count as correct iff there is no interpretation which makes all the formulae in Γ true and all the formulae in Δ false. Since Γ and Δ are both constrained to be finite, this comes to the same thing as saying that the conjunction of all the formulae in Γ entails the disjunction of all the formulae in Δ. (For this purpose we may, if we wish, take the 'empty conjunction' as the formula $\top$ and the 'empty disjunction' as the formula $\bot$.) So such a sequent is always equivalent to one with just one formula on either side. That, of course, always was the case with the sequents we have considered previously. But just as previously we could set out our rules with several formulae on the left, without needing occurrences of $\wedge$ to bind them together into one, so now we can do the same on the right as well, without binding the several formulae together by occurrences of $\vee$. This restores the symmetry between $\wedge$ and $\vee$ that was so clearly missing in the natural deduction approach, and it gives us a great freedom to formulate elegant rules for the truth-functors and quantifiers, as we shall see. But as a preliminary let us first notice the structural rules for a sequent calculus employing these new Gentzen sequents.

So far as the standard rules are concerned, Assumptions will remain as before, Thinning and Interchange and Contraction will be extended so that they apply to both sides of a sequent, and Cut will be reformulated in a more powerful way, suited to the more complex sequents now available. That is to say, the standard structural rules are now these:

$$(\text{ASS})\frac{}{\varphi \Rightarrow \varphi}$$

$$(\text{THIN})\frac{\Gamma \Rightarrow \Delta}{\Gamma,\varphi \Rightarrow \Delta} \qquad (\text{THIN})\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi,\Delta}$$

$$(\text{CUT})\frac{\Gamma_1 \Rightarrow \varphi,\Delta_1 \quad \Gamma_2,\varphi \Rightarrow \Delta_2}{\Gamma_1,\Gamma_2 \Rightarrow \Delta_1,\Delta_2}$$

$$(\text{INT})\frac{\Gamma,\varphi,\psi,\Delta \Rightarrow \Theta}{\Gamma,\psi,\varphi,\Delta \Rightarrow \Theta} \qquad (\text{INT})\frac{\Gamma \Rightarrow \Delta,\varphi,\psi,\Theta}{\Gamma \Rightarrow \Delta,\psi,\varphi,\Theta}$$

$$(\text{CONTR})\frac{\Gamma,\varphi,\varphi \Rightarrow \Delta}{\Gamma,\varphi \Rightarrow \Delta} \qquad (\text{CONTR})\frac{\Gamma \Rightarrow \varphi,\varphi,\Delta}{\Gamma \Rightarrow \varphi,\Delta}$$

If we wish to adopt a rule for basic sequents, in place of Assumptions and Thinning, then that rule must also be extended as Thinning has been extended, i.e. to

$$(\text{BS})\frac{}{\Gamma,\varphi \Rightarrow \varphi,\Delta}.$$

These are the structural rules that one *expects* to find holding in a Gentzen sequent calculus, either adopted as primitive rules of the system or derived from other rules. (For example, thinning on the left is derivable from the natural deduction rules for $\wedge$, as we noted long ago; thinning on the right will now be derivable from the symmetrical rules for $\vee$; cutting may well be derivable from the rules for $\rightarrow$, depending on just what rules are adopted here.) But in particular cases a calculus may be specified which lacks one or more of these rules. However, all the calculi that will be considered here will contain Interchange as a primitive rule, and to avoid clutter I shall continue to leave the applications of this rule tacit.

I briefly illustrate the new freedoms with a few examples. The standard natural deduction rules for $\wedge$ are formulated as rules for introducing and eliminating on the right, and we could already have formulated a similar pair of rules for $\vee$, for introducing and eliminating on the left. But the duality of these rules can now be brought out much more clearly. To obtain a succinct formulation, let us again use a double horizontal line to signify that the inference holds both from top to bottom and from bottom to top. Then these rules are

$$\begin{array}{c} \Gamma \Rightarrow \varphi,\Delta \quad \text{AND} \quad \Gamma \Rightarrow \psi,\Delta \\ \hline\hline \Gamma \Rightarrow \varphi\wedge\psi,\Delta \end{array} \qquad \begin{array}{c} \Gamma,\varphi \Rightarrow \Delta \quad \text{AND} \quad \Gamma,\psi \Rightarrow \Delta \\ \hline\hline \Gamma,\varphi\vee\psi \Rightarrow \Delta \end{array}$$

(Previously we had to require Δ to be null in the rule for $\wedge$, and to be a single formula in the rule for $\vee$, and this destroyed the symmetry.) A more significant improvement, however, is that we can now give a much simpler pair of rules for $\wedge$, which introduce it and eliminate it on the left, and can match these with an equally simple pair of rules for $\vee$, which introduce it and eliminate it on the right:

$$\begin{array}{c} \Gamma,\varphi,\psi \Rightarrow \Delta \\ \hline\hline \Gamma,\varphi\wedge\psi \Rightarrow \Delta \end{array} \qquad \begin{array}{c} \Gamma \Rightarrow \varphi,\psi,\Delta \\ \hline\hline \Gamma \Rightarrow \varphi\vee\psi,\Delta \end{array}$$

As is familiar, the rules for $\wedge$ may be reformulated once more in this even simpler way:

$$\varphi,\psi \Rightarrow \varphi\wedge\psi \qquad \varphi\wedge\psi \Rightarrow \varphi, \quad \varphi\wedge\psi \Rightarrow \psi.$$

And these rules too can now be matched by dual rules for $\vee$:

$$\varphi\vee\psi \Rightarrow \varphi,\psi \qquad \varphi \Rightarrow \varphi\vee\psi, \quad \psi \Rightarrow \varphi\vee\psi.$$

As you are invited to discover, all these various ways of framing rules for $\wedge$ and for $\vee$ are equivalent to one another, given the standard structural rules in the background.

The situation with $\rightarrow$ is similarly improved, as Exercises 7.4.3 and 7.4.5 will show. But perhaps the most welcome liberation comes with the rules for $\neg$, for the pair TND and EFQ can now be put in this simple way:

$$\Rightarrow \varphi,\neg\varphi \qquad \varphi,\neg\varphi \Rightarrow.$$

These rules are adequate by themselves. So also would be either of these pairs of rules, the first for introducing and eliminating on the left, and the second for introducing and eliminating on the right

$$\begin{array}{c} \Gamma \Rightarrow \varphi,\Delta \\ \hline\hline \Gamma,\neg\varphi \Rightarrow \Delta \end{array} \qquad \begin{array}{c} \Gamma,\varphi \Rightarrow \Delta \\ \hline\hline \Gamma \Rightarrow \neg\varphi,\Delta \end{array}$$

While we had to require Δ to be empty, we had the oddity that the pair on the left was *not* adequate, though the pair on the right was adequate. (See Exercises 7.2.2(*b*) and 7.4.3.) This, I hope, is sufficient illustration of how the new style of sequent allows us much more freedom in the formulation of rules for truth-functors, and a considerable increase in elegance. But let us now come back to the question with which this section began, of how to improve our formulation of the tableau rules as a sequent calculus.

Our first formulation of the method of semantic tableaux in Sections 4.1–2 made overt use of semantical vocabulary, with formulae being explicitly assigned a truth-value, T or F. This was clumsy in practice, so in Section 4.3 we introduced an equivalent but abbreviated version, which eliminated the semantical vocabulary, but at the cost of giving a special role to negation. Let us now return to the original version, which may be somewhat long-winded but is also very much more elegant, as we noted at the time. In the original version truth and falsehood are symmetrically treated, and there is no special role for negation. How, then, should we formulate suitable sequent calculus rules to fit the original version of the semantic tableaux?

At the root of the tableau we have a set of formulae, some assigned the value T and some assigned the value F. This represents the hypothesis that truth-values can indeed be assigned as indicated. But if the proof is successful, it shows that this hypothesis runs into a contradiction, and hence that truth-values cannot be assigned as indicated. Now suppose we write 'on the left' all those formulae assigned the value T in the root, and 'on the right' all those assigned the value F. Then what is proved is that there is no interpretation which gives T to all those on the left and F to all those on the right. In other words, what is proved is the Gentzen sequent which has on its left all the formulae assigned T in the root and on the right all the formulae assigned F in the root. And it is not just the result of the whole proof that can be seen in this way, for indeed each step of the proof can be seen as reasoning about Gentzen sequents. We begin with the hypothesis that a certain Gentzen sequent is not correct, and the steps of developing this hypothesis are inferences that in that case certain further sequents are not correct either. The case of negation provides a convenient example. Suppose that our hypothesis so far is that a formula $\neg\varphi$ is true, that certain other formulae Γ are all true, and that other formulae Δ are all false. Applying the rule for negation then represents this inference:

Suppose $\Gamma,\neg\varphi \not\Rightarrow \Delta$
Then $\Gamma \not\Rightarrow \varphi,\Delta$.

Similarly, if our hypothesis had been that $\neg\varphi$ is false, then applying the negation rule would be inferring thus:

Suppose $\Gamma \not\Rightarrow \neg\varphi,\Delta$
Then $\Gamma,\varphi \not\Rightarrow \Delta$.

(Notice that, as in Section 7.3, we have deleted from these inferences the superfluous repetition of $\neg\varphi$.) It is clear that all the original tableau rules can be rephrased in this way.

When we do reformulate all the rules thus, and then turn them upside-down so that they become rules of a standard sequent calculus, the result is this:

$$(\text{BS})\ \frac{}{\Gamma,\varphi \Rightarrow \varphi,\Delta}$$

$$(\wedge\Rightarrow)\ \frac{\Gamma,\varphi,\psi \Rightarrow \Delta}{\Gamma,\varphi\wedge\psi \Rightarrow \Delta} \qquad (\Rightarrow\wedge)\ \frac{\Gamma \Rightarrow \varphi,\Delta \quad \Gamma \Rightarrow \psi,\Delta}{\Gamma \Rightarrow \varphi\wedge\psi,\Delta}$$

$$(\vee\Rightarrow)\ \frac{\Gamma,\varphi \Rightarrow \Delta \quad \Gamma,\psi \Rightarrow \Delta}{\Gamma,\varphi\vee\psi \Rightarrow \Delta} \qquad (\Rightarrow\vee)\ \frac{\Gamma \Rightarrow \varphi,\psi,\Delta}{\Gamma \Rightarrow \varphi\vee\psi,\Delta}$$

$$(\rightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \varphi,\Delta \quad \Gamma,\psi \Rightarrow \Delta}{\Gamma,\varphi\rightarrow\psi \Rightarrow \Delta} \qquad (\Rightarrow\rightarrow)\ \frac{\Gamma,\varphi \Rightarrow \psi,\Delta}{\Gamma \Rightarrow \varphi\rightarrow\psi,\Delta}$$

$$(\neg\Rightarrow)\ \frac{\Gamma \Rightarrow \varphi,\Delta}{\Gamma,\neg\varphi \Rightarrow \Delta} \qquad (\Rightarrow\neg)\ \frac{\Gamma,\varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi,\Delta}$$

$$(\forall\Rightarrow)\ \frac{\Gamma,\varphi(\alpha/\xi) \Rightarrow \Delta}{\Gamma,\forall\xi\varphi \Rightarrow \Delta} \qquad (\Rightarrow\forall)\ \frac{\Gamma \Rightarrow \varphi,\Delta}{\Gamma \Rightarrow \forall\xi\varphi(\xi/\alpha),\Delta}$$
provided α is not in Γ or Δ

$$(\exists\Rightarrow)\ \frac{\Gamma,\varphi \Rightarrow \Delta}{\Gamma,\exists\xi\varphi(\xi/\alpha) \Rightarrow \Delta} \qquad (\Rightarrow\exists)\ \frac{\Gamma \Rightarrow \varphi(\alpha/\xi),\Delta}{\Gamma \Rightarrow \exists\xi\varphi,\Delta}$$
provided α is not in Γ or Δ

As before, all our rules are introduction rules, but they now pair nicely into rules for introducing on the left, labelled ($*\Rightarrow$), and on the right, labelled ($\Rightarrow*$), for each truth-functor or quantifier $*$. Also, the negation sign is no longer playing any special role, but occurs only in the pair of rules that deal with it. This calculus, then, represents in a much nicer way the principles that are at work in a tableau proof. As before we do not have the structural rules ASS and THIN, but instead the rule BS, which does the same work. Also, as before we do not have the rule CUT, since tableau proofs do not use any such rule. For this reason, the calculus is known as Gentzen's cut-free sequent calculus. Finally, the rules do include INT, if we need to state that rule separately, and as formulated here they need to include CONTR. But, as in Exercise 7.3.2, we could avoid this by reformulating ($\forall\Rightarrow$) and ($\Rightarrow\exists$) in this way:

$$(\forall\Rightarrow')\ \frac{\Gamma,\forall\xi\varphi,\varphi(\alpha/\xi)\Rightarrow\Delta}{\Gamma,\forall\xi\varphi\Rightarrow\Delta} \qquad (\exists\Rightarrow')\ \frac{\Gamma\Rightarrow\varphi(\alpha/\xi),\exists\xi\varphi,\Delta}{\Gamma\Rightarrow\exists\xi\varphi,\Delta}$$

Given this reformulation, and for completeness adding INT explicitly, the rules stated here exactly match the rules of the original tableau system, so it is easy to argue that whatever sequents can be proved in the one system can also be proved in the other.

EXERCISES

7.4.1. Rewrite the proof, given on p. 162, of the sequent

$$\forall x\exists y(Fx\wedge Gy)\Rightarrow\exists y\forall x(Fx\wedge Gy)$$

as a proof in Gentzen's cut-free sequent calculus.

7.4.2. Assuming all the standard rules for a calculus of Gentzen sequents, verify the assertion made in the text, that the various sets of rules cited for $\wedge$ and for $\vee$ on p. 294 are interdeducible.

7.4.3. Assuming all the standard structural rules, show that the Gentzen rules ($\rightarrow\Rightarrow$) and ($\Rightarrow\rightarrow$) are interdeducible with each of the following sets:

$$(a)\ \frac{\Gamma\Rightarrow\varphi,\Delta \quad \text{AND} \quad \Gamma,\psi\Rightarrow\Delta}{\Gamma,\varphi\rightarrow\psi\Rightarrow\Delta}$$

$$(b)\ \frac{\Gamma,\varphi\Rightarrow\psi,\Delta}{\Gamma\Rightarrow\varphi\rightarrow\psi,\Delta}$$

$$(c)\ \psi\Rightarrow\varphi\rightarrow\psi \qquad \Rightarrow\varphi,\varphi\rightarrow\psi \qquad \varphi,\varphi\rightarrow\psi\Rightarrow\psi$$

7.4.4. Assuming all the standard structural rules, consider this pair of rules for negation:

$$\frac{\Gamma,\varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi,\Delta}.$$

(*a*) Suppose first that in these rules Δ is required to be empty. Show that in that case the sequent $\neg\neg P \Rightarrow P$ is not provable. [Method: consider this three-valued table for negation:

φ	$\neg\varphi$
1	0
½	0
0	1

(Compare table VI on p. 198.) Count a sequent $\Gamma \Rightarrow \psi$ as correct iff the minimum of the values of the formulae in Γ is less than or equal to the value of ψ. Verify that on this interpretation the structural rules remain sound, and the two rules for ¬ are both sound, but the proposed sequent is not correct.]

(*b*) Allowing Δ to be non-empty, prove the sequent $\neg\neg P \Rightarrow P$. [Hint: you will find it useful to use CUT on $\Rightarrow P,\neg P$ and $\neg P \Rightarrow \neg\neg\neg P$.]

(*c*) Show that the pair of rules in question is equivalent to the Gentzen pair (¬⇒) and (⇒¬). [For the argument in one direction you will need part (*b*); for the other direction you will need $P \Rightarrow \neg\neg P$.]

7.4.5. (This exercise continues Exercise 5.7.2.) Let GC be a sequent calculus for Gentzen sequents whose only truth-functor is →. It has the standard structural rules and in addition just (→⇒) and (⇒→).

(*a*) Show that (⇒→) can equivalently be replaced by the pair of rules

$$\frac{\Gamma,\varphi \Rightarrow \Delta}{\Gamma \Rightarrow \varphi\rightarrow\psi,\Delta} \qquad \frac{\Gamma \Rightarrow \psi,\Delta}{\Gamma \Rightarrow \varphi\rightarrow\psi,\Delta}$$

(*b*) Let φ be a formula with → as its only truth-functor, and consider any assignment of truth-values to the letters in φ. Let Γ be the set of letters assigned T, and Δ be the set of letters assigned F. Prove:

If φ is true on this assignment, then $\Gamma \Rightarrow \varphi,\Delta$ is provable in GC.
If φ is false on this assignment, then $\Gamma,\varphi \Rightarrow \Delta$ is provable in GC.

[Method: use induction on the length of φ.]

(*c*) Deduce from (*b*) that if φ is a tautology, with → as its only truth-functor, then $\Rightarrow \varphi$ is provable in GC.

(*d*) Deduce from (*c*) that if $\Gamma \models \Delta$, and if → is the only truth-functor in Γ and in Δ, then $\Gamma \Rightarrow \Delta$ is provable in GC. [Hints: (1) you can define ∨ in terms of →; (2) you will need to derive the following two further rules of GC:

$$\frac{\Gamma \Rightarrow \varphi \rightarrow \psi, \Delta}{\Gamma, \varphi \Rightarrow \psi, \Delta} \qquad \frac{\Gamma \Rightarrow (\varphi \rightarrow \psi) \rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi, \psi, \Delta}$$

Note, incidentally, that a rather quicker proof of this same result is contained in the reasoning that immediately follows.]