EXERCISE 55. Show that the following will be evaluated as true given any interpretation of the sentence constants. Warning: the question is about what semantic values these expressions will have. If you want to make use of some proof or derivation system you've learned for sentential logic, that may be acceptable, but only if you also explain what relevance provability in that system has to the question that was explicitly asked. Another strategy is to answer these questions using arguments like you used to answer exercise 54.

a. $(P \supset Q) \vee (Q \supset P)$

b. $P \vee (P \supset Q)$

c. $((P \supset Q) \supset P) \supset P$

This last sentence is an instance of what's known as Peirce's Law. It may look ugly but in fact marks an important dividing line between classical logic and some weaker logics, in some ways more important than $\neg\neg P \supset P$ or $P \vee \neg P$

Here Are 2 sample solutions for 55a.

55. Each of these solutions is of the same general shape. We want to prove that given any interpretation of the sentence constants, the expression in question will be evaluated as true. So our goal for each expression will be to "reduce" it to a specification of the interpretation of its atoms, and then to show that this specification will be met by any interpretation.

a. Let P and Q be arbitrary sentence constants. We have established that $V_I(\phi\vee\psi) = 1$ iff V_I(\phi) = 1 or V_I(\psi) = 1$. So (P\horseshoe Q)\vee(Q\horseshoe P) is true iff $V_I(P\horseshoe Q) = 1 or V_I(Q\horseshoe P) = 1$. By the definition of implication and the associativity of disjunction in our metalanguage, it follows that this sentence is true if one of the following disjuncts is satisfied: $V_I(P) = 0 or V_I(Q) = 1 or V_I(P) = 1 or V_I(Q) = 0$. Any interpration of P and Q will satisfy at least two (actually, exactly two) of these disjuncts.

Homework Exercise 55

(a) Take an arbitrary interpretation I. Assume that $V_I((P \rightarrow Q) \vee (Q \rightarrow P))=0$. This implies that it is not the case that ($V_I(P \rightarrow Q)=1$ or $V_I(Q \rightarrow P)=1$), i.e. it is not the case that $V_I(P \rightarrow Q)=1$ and it is not the case that $V_I(Q \rightarrow P)=1$. The former implies that $V_I(P \rightarrow Q)=0$ and the latter that $V_I(Q \rightarrow P)=0$. Now, $V_I(P \rightarrow Q)=0$ implies that it is not the case that ($V_I(P)=0$ or $V_I(Q)=1$). That is, it is not the case that $V_I(P)=0$ and it is not the case that $V_I(Q)=1$, i.e. $V_I(P)=1$ and $V_I(Q)=0$. $V_I(Q \rightarrow P)=0$ implies that it is not the case that ($V_I(Q)=0$ or $V_I(P)=1$). That is, it is not the case that $V_I(Q)=0$ and it is not the case that $V_I(P)=1$, i.e. $V_I(Q)=1$ and $V_I(P)=0$. Putting things together, we have that $V_I(P)=1$ and that $V_I(P)=0$. Contradiction! Thus, $V_I((P \rightarrow Q) \vee (Q \rightarrow P))=1$, and since I was arbitrary, we have our result.

EXERCISE 56. Show that these are logically/semantically equivalent. See the comments on Exercise 55 above.

a. $P \vee Q$, $(P \supset Q) \supset Q$

b. $\neg P$, $P \supset \bot$

c. $P \supset (Q \vee R)$, $(P \supset Q) \vee (P \supset R)$

d. $(Q \vee R) \supset P$, $(Q \supset P) \vee (R \supset P)$

e. $P \supset (Q \supset R)$, $Q \supset (P \supset R)$, $(P \wedge Q) \supset R$

f. $P \supset \neg Q$, $Q \supset \neg P$

Here are 3 sample solutions for 56c

c. $P \horseshoe (Q \lor R)$ is true just in case either P is false or $Q \lor R$ is true (i.e., in case either P is false or Q is true or R is true). $(P \horseshoe Q) \lor (P \horseshoe R)$ is true just in case either $P \horseshoe Q$ is true or $P \horseshoe R$ is true. It follows that $(P \horseshoe Q) \lor (P \horseshoe R)$ is true just in case either (P is false or Q is true) or (P is false or R is true) (i.e., just in case either P is false or Q is true or R is true). Thus, both $P \horseshoe (Q \lor R)$ and $(P \horseshoe Q) \lor (P \horseshoe R)$ are true just in case either P is false or Q is true or R is true.

```
c. V(P\horseshoe(Q\vee R))=1 iff V(P)=0 or V(Q\vee R)=1 (t.c. of
\horseshoe)
   V(Q\vee R)=1 iff V(Q)=1 or V(R)=1 (t.c. of \vee)
   So this formula is evaluated 1 iff V(Q)=1,or V(R)=1, or
V(P)=0.

   V((P\horseshoe Q)\vee(P\horseshoe R))=1 iff V(P\horseshoe
Q)=1 or V(P\horseshoe R)=1 (t.c. of \vee)
   V(P\horseshoe Q)=1 iff V(P)=0 or V(Q)=1 (t.c. of \horseshoe)
   V(P\horseshoe R)=1 iff V(P)=0 or V(R)=1 (t.c. of \horseshoe)
   So this whole formula is true iff V(P)=0, or V(Q)=1, or
V(R)=1.
```

This solution is ok but would be easier to read with better signposting.

```
(c) Take an arbitrary interpretation $I$. Suppose that $V_I(P
\rightarrow (Q \vee R))=1$. This implies that $V_I(P)=0$ or $V_I(Q
\vee R)=1$. If the former, then $V_I(P)=0$ or $V_I(Q)=1$ holds, i.e.
$V_I(P \rightarrow Q)=1$ holds. This implies that $V_I(P \rightarrow
Q)=1$ or $V_I(P \rightarrow R)=1$ holds, i.e. $V_I((P \rightarrow Q)
\vee (P \rightarrow R))=1$. Now suppose that $V_I(Q \vee R)=1$. This
implies that $V_I(Q)=1$ or $V_I(R)=1$. Suppose that $V_I(Q)=1$. Then
$V_I(P)=0$ or $V_I(Q)=1$, i.e. $V_I(P \rightarrow Q)=1$. This implies
that $V_I(P \rightarrow Q)=1$ or $V_I(P \rightarrow R)=1$ holds, i.e.
$V_I((P \rightarrow Q) \vee (P \rightarrow R))=1$. Now suppose that
$V_I(R)=1$. Then $V_I(P)=0$ or $V_I(R)=1$, i.e. $V_I(P \rightarrow
R)=1$. This implies that $V_I(P \rightarrow Q)=1$ or $V_I(P
\rightarrow R)=1$ holds, i.e. $V_I((P \rightarrow Q) \vee (P
\rightarrow R))=1$. Thus, in any case, $V_I((P \rightarrow Q) \vee (P
\rightarrow R))=1$.
```

This shows $P \supset Q \vee R \vDash (P \supset Q) \vee (P \supset R)$

```
Now suppose that $V_I((P \rightarrow Q) \vee (P \rightarrow R))=1$.
This implies that $V_I(P \rightarrow Q)=1$ or $V_I(P \rightarrow
R)=1$. Suppose that $V_I(P \rightarrow Q)=1$. Then $V_I(P)=0$ or
$V_I(Q)=1$. This implies that $V_I(P)=0$ or ($V_I(Q)=1$ or
$V_I(R)=1$), i.e. $V_I(P)=0$ or $V_I(Q \vee R)=1$, i.e. $V_I(P
\rightarrow (Q \vee R))=1$. Now suppose that $V_I(P \rightarrow R)=1$.
Then $V_I(P)=0$ or $V_I(R)=1$. This implies that $V_I(P)=0$ or
($V_I(Q)=1$ or $V_I(R)=1$), i.e. $V_I(P)=0$ or $V_I(Q \vec R)=1$, i.e.
$V_I(P \rightarrow (Q \vee R))=1$. Thus, in any event $V_I(P
\rightarrow (Q \vee R))=1$.
```

This shows $(P \supset Q) \vee (P \supset R) \vDash P \supset Q \vee R$

EXERCISE 57. Show that these are logically/semantically equivalent. See the comments on Exercise 55 above.

a. $\forall x(Fx \supset P), \exists xFx \supset P$

b. $P \supset \forall xFx, \forall x(P \supset Fx)$

c. $\forall x\neg Fx, \neg\exists xFx$

d. $\forall x(Fx \wedge Gx), \forall xFx \wedge \forall xGx$

e. $\exists x(Fx \vee Gx), \exists xFx \vee \exists xGx$

Here are some sample solutions for 57a.

Goes a bit faster than you should go.

(a) Suppose for some V_g that $V_g(\forall x(Fx \supset P)) = 1$. Then, for every $u \in \mathcal{D}$, $V_{\mathcal{M},g^x_u}(Fx \supset P) = 1$. Therefore, for every $u \in \mathcal{D}$, either $V_{\mathcal{M},g^x_u}(Fx) = 0$ or $V_{\mathcal{M},g^x_u}(P) = 1$. Assume for some arbitrary $u \in \mathcal{D}$, $V_{\mathcal{M},g^x_u}(Fx) = 1$. It must then be the case that $V_{\mathcal{M},g^x_u}(P) = 1$. Therefore, $V_g(\exists xFx \supset P)$ would $= 1$. Now assume that for *no* arbitrary $u \in \mathcal{D}$, $V_{\mathcal{M},g^x_u}(Fx) = 1$. Then $V_g(\exists xFx \supset P) = 1$ because the antecedent is false on all assignment functions.

This shows that $\forall x(Fx \supset P) \vDash \exists xFx \supset P$

Suppose for some V_g that $V_g(\exists xFx \supset P) = 1$. Then, either $V_g(\exists xFx) = 0$ or $V_g(P) = 1$. Assume $V_g(P) = 1$. Then, by the definition of $\supset$, $V_g(\forall x(Fx \supset P)) = 1$. Assume $V_g(P) = 0$. Then $V_g(\exists xFx) = 0$. So $V_g(\sim \exists xFx) = 1$. So $V_g(\forall x{\sim}Fx) = 1$. And therefore, by the definition of $\supset$ $V_g(\forall x(Fx \supset P)) = 1$. ← Be more explicit about why.

This shows that $\exists xFx \supset P \vDash \forall x(Fx \supset P)$

This is correct, but too much is compressed into a single step here, and here.

Yes, but be more explicit about why.

To show equivalence, you should show that for every model M and assignment g the left-hand side (lhs) is true on M, g iff the rhs is too. This starts out just talking about the mere truth of the lhs – omitting M and g. If we had specified g for the lhs, then when we come to the rhs we wouldn't be interested in every assignment, but only those that differ from g at most wrt x.

57.

a. $\forall x (Fx \horseshoe P)$ is true just in case $V_{M,g}(Fx \horseshoe P)$ = 1 for any variable assignment g. By definition of $\horseshoe$, $V_{M,g}(Fx \horseshoe P)$ = 1 just in case $V_{M,g}(Fx)$ = 0 or $V_{M,g}(P)$ = 1. By definition of a model, $V_{M,g}(Fx)$ = 0 just in case the interpretation function of M does not assign F to (the variable assignment of) x in M- that is, if x is not in the extension of F under M and g. So, $\forall x (Fx \horseshoe P)$ is true just in case for any variable assignment g, either $V_{M,g}(Fx)$ = 0 or $V_{M,g}(P)$ = 1.

$\exists x Fx \horseshoe P$ is true just in case (by definition of $\horseshoe$) either $\exists x Fx$ = 0 or P = 1. P = 1 just in case $V_{M,g}(P)$ = 1. $\exists x Fx$ = 0 just in case $V_{M,g}(Fx)$ = 0 for every variable assignment g. $V_{M,g}(Fx)$ = 0 just in case the interpretation function of M does not assign F to (the variable assignment of) x in M- that is, if x is not in the extension of F under M and g. So, $\exists x Fx \horseshoe P$ is true just in case for any variable assignment g, either $V_{M,g}(Fx)$ = 0 or $V_{M,g}(P)$ = 1.

Similar problem here. It pays to be explicit about your quantification over Ms and gs, to be sure you're doing it at the right place.

So, the two formulas are equivalent.

Some more solutions for 57a.

EXERCISE 57.

a) Suppose, per impossibile, that there is a model $\mathcal{M}$ and some variable assignment g for $\mathcal{M}$ such that, (i) $V_{\mathcal{M}g}(\forall x(Fx \supset P)) = 1$ and (ii) $V_{\mathcal{M}g}(\exists xFx \supset P)) = 0$. (Henceforth I drop the $\mathcal{M}$ subscript for the valuation function V in all exercises). From (ii) $V_g(\exists xFx) = 1$ and $V_g(P) = 0$. So for some $u \in \mathcal{D}$, call one of them "a", (iii) $V_{g^x_a}(\cancel{\exists x}(Fx)) = 1$ and $V_{g^x_a}(P) = 0$. Thus $[x]_{g^x_a} \in \mathcal{I}(F)$. so (iv) $\langle a\rangle \in \mathcal{I}(F)$. Now, from (i) we get, for all $u \in \mathcal{D}$, $V_{g^x_u}(Fx \supset P) = 1$. Since $[a]_g \in \mathcal{D}$, in particular we get $V_{g^x_a}(Fx \supset P) = 1$. But this means that (v) either $V_{g^x_a}(Fx) = 0$ or $V_{g^x_a}(P) = 1$, but the latter disjunct is impossible given the second conjunct of (iii), so the former disjunct must hold. This disjunct entails $[x]_{g^x_a} \notin \mathcal{I}(F)$. But this contradicts (iv). This shows that $\forall x(Fx \supset P) \vDash \exists xFx \supset P$.

∃x should be omitted. typo?

Suppose, per impossibile, that there is a model $\mathcal{M}$ and some variable assignment g for $\mathcal{M}$ such that, (i) $V_g(\forall x(Fx \supset P)) = 0$ and (ii) $V_g(\exists xFx \supset P)) = 1$. From (ii) we get (iii), that either $V_g(\exists xFx)) = 0$ or $V_g(P)) = 1$. From (i) it follows that there is some $u \in \mathcal{D}$, call it "a", such that $V_{g^x_a}(Fx \supset P) = 0$, i.e. that $V_{g^x_a}(Fx) = 1$ and $V_{g^x_a}(P) = 0$. But this entails that (iv) $V_g(\exists xFx) = 1$ and $V_g(P) = 0$. (iv) is impossible given (iii). This shows that $\exists xFx \supset P \vDash \forall x(Fx \supset P)$.

This shows that there is no model and variable assignment (since all formulae are closed, the results proved on variable assignment g hold for all variable assignments) in which the formulae under consideration differ in truth value and thus shows that the formulae are semantically equivalent.

57. In this exercise, like in 56, we'll show that the expressions in question have the same truth conditions.

a. $V_g(\forall x(Fx\supset P))=1$ iff for every u \in D, V_{g[x:=u]}(Fx\horseshoe P)=1$. This, in turn, is true just in case for every u \in D, V_{g[x:=u]}(Fx)=0 or V_{g[x:=u]}(P)=1. In other words, for every u in D, either I(u) \notin I(F) or (because there are no instances of x in P) I(P)=1.

$\boxed{I(u)}$ isn't defined – you just mean $\boxed{u}$.

$\exists xFx\supset P$ is equivalent to $\neg\forall x\neg Fx \supset P$. Note that P here is not in the scope of the quantifier or the negation under the quantifier. This new expression <u>is true</u> iff for every u\in D, $V_{g[x:=u]}(\neg Fx)=0$ (equivalently by the definition of negation: V_{g[x:=u]}(Fx)=1$) or $V_{g[x:=u]}(P)=1$. In other words, for every u \in D, either I(u)\in I(F), or I(P)=0.

In what model and assignment?

These turn out not to be the same truth conditions: the first expression is true just in case either every entity in the domain is not in F, or P is true. The second is true just in case either every entity in the domain **is** in F, or P is true.

EXERCISE 58. Show that these are not equivalent.

a. $\forall x(Fx \vee Gx)$, $\forall xFx \vee \forall xGx$

b. $\exists x(Fx \wedge Gx)$. $\exists xFx \wedge \exists xGx$

SAmple solutions for 58a

58. (a) Domain = {a, b, c, d}; Interpretation I: F = {a, b}, G = {c, d}. All assignments of u to x where u ∈ Domain make ∀x(Fx ∨ Gx) true, but none make ∀xFx ∨ ∀xGx true.

Seems perfectly Adequate.

Here's A more explicit Answer.

EXERCISE 58.

a) $\mathcal{D} = \{u, v\}$

$\mathcal{I}(F) = \{u\}$

$\mathcal{I}(G) = \{v\}$

Take an arbitrary variable assignment g. $V_{g^u_x}(Fx) = 1$, and hence $V_{g^u_x}(Fx) = 1$ or $V_{g^u_x}(Gx) = 1$. Thus $V_{g^u_x}(Fx \vee Gx) = 1$. Moreover, $V_{g^v_x}(Gx) = 1$, and so $V_{g^v_x}(Fx) = 1$ or $V_{g^v_x}(Gx) = 1$. Thus $V_{g^v_x}(Fx \vee Gx) = 1$. But this exhausts our domain, and hence for all $o \in \mathcal{D}$, $V_{g^o_x}(Fx \vee Gx) = 1$, that is, $V_g(\forall x(Fx \vee Gx)) = 1$. However, $V_g(\forall xFx) = 0$, since $V_{g^v_x}(Fx) = 0$. Furthermore, $V_g(\forall xGx) = 0$, since

$V_{g^u_x}(Gx) = 0$. Hence $V_g(\forall xFx \vee \forall xGx) = 0$.

EXERCISE 59. Do Sider's exercise 4.1 on p. 96. (There's a hint in Appendix A.)

Exercise 4.1** Show that if ϕ has no free variables, then for any model $\mathscr{M}$ and variable assignments g and h for $\mathscr{M}$, $V_{\mathscr{M},g}(\phi)=V_{\mathscr{M},h}(\phi)$

Call this thesis 1.

Exercise 4.1 Hint: first prove by induction that for any wff ϕ and mode
variable assignments g and h agree on all variables with free occurrences in
$V_{\mathscr{M},g}(\phi)=V_{\mathscr{M},h}(\phi)$; and then use this fact to establish the desired result.

Call this thesis H (for "hint").

It would be difficult to Argue inductively directly for thesis 1, but it is easy to Argue inductively for thesis H, and then derive thesis 1 from that.

Here is A sample solution.

Homework Exercise 59

Proposition 1: For any wff ψ and model M, if variable assignments g and h agree on all variables with free occurrences in ψ, then $V_{M,g}(\psi)=V_{M,h}(\psi)$.

What I called thesis H.

Proof: We proceed by induction on the complexity of ψ.

Inductive base: ψ is composed of an n-place predicate (P), and n occurrences of terms ($a_1, a_2, ..., a_n$). Choose an arbitrary model M, and variable assignments g and h. By assumption, g and h agree on the free variables which occur in ψ. Those terms which are not free variables are constants, and the denotation of constants is determined by M, which remains fixed. Thus, $[a_i]_{M,g}=[a_i]_{M,h}$, for $1 \leq i \leq n$. Thus, $V_{M,g}(\psi)=1$ iff $\langle [a_1]_{M,g}, [a_2]_{M,g},...,[a_n]_{M,g} \rangle \in I(P)$ iff $\langle [a_1]_{M,h}, [a_2]_{M,h},...,[a_n]_{M,h} \rangle \in I(P)$ iff $V_{M,h}(\psi)=1$.

Inductive step: Assume that the hypothesis holds for all formulas χ. Let ψ = $\neg \chi$. Choose an arbitrary model M, and variable assignments g and h such that g and h agree on the free variables which occur in ψ. Then $V_{M,g}(\psi)=1$ iff $V_{M,g}(\neg \chi)=1$ iff $V_{M,g}(\chi)=0$ iff $V_{M,h}(\chi)=0$ (by assumption) iff $V_{M,g}(\neg \chi)=1$ iff $V_{M,h}(\psi)=1$.

Continued ↓

Continuing the solution to exercise 59 (Sider's 4.1)

Assume that the hypothesis holds for all formulas χ, γ. Let $\psi = \chi \rightarrow \gamma$. Choose an arbitrary model M, and variable assignments g and h. By assumption, g and h agree on the free variables which occur in ψ. Then $V_{M,g}(\psi)=1$ iff $V_{M,g}(\chi \rightarrow \gamma)=1$ iff $V_{M,g}(\chi)=0$ or $V_{M,g}(\gamma)=1$. Now, $V_{M,g}(\chi)=0$ iff $V_{M,h}(\chi)=0$ (by assumption) iff $V_{M,h}(\chi)=0$ or $V_{M,h}(\gamma)=1$. And $V_{M,g}(\gamma)=1$ iff $V_{M,h}(\gamma)=1$ (by assumption) iff $V_{M,h}(\chi)=0$ or $V_{M,h}(\gamma)=1$. Thus, $V_{M,g}(\chi)=0$ or $V_{M,g}(\gamma)=1$ iff $V_{M,h}(\chi)=0$ or $V_{M,h}(\gamma)=1$ iff $V_{M,h}(\chi \rightarrow \gamma)=1$ iff $V_{M,h}(\psi)=1$.

Assume that the hypothesis holds for all formulas χ. Let $\psi = \forall x \chi$. Choose an arbitrary model M, and variable assignments g and h. By assumption, g and h agree on the free variables which occur in ψ. Then $V_{M,g}(\psi)=1$ iff $V_{M,g}(\forall x \chi)$ iff for all $s \in D, V_{M,g_{s}^{x}}(\chi)=1$ iff for all $s \in D, V_{M,h_{s}^{x}} (\chi)=1$ (since, for all $s \in D, g_{s}^{x}$ and h_{s}^{x} agree on the free variables in χ, and we're assuming that the hypothesis holds for χ) iff $V_{M,h}(\forall x \chi)=1$ iff $V_{M,h}(\psi)=1$. This completes the proof.

This is where the Action is. There's A bit of trickiness here. The Author hasn't really shown what we need. He or she has shown that for all g, h that Agree wrt free variables in χ, they will give same semantic value to $\forall x \chi$.

But we need to show this when g, h merely Agree wrt the free variables in $\forall x \chi$ (those may be fewer free variables, since x is now bound.)

Proposition 2: For any model M, variable assignments g, h, and formula ψ, if ψ has no free variables, then $V_{M,g}(\psi)= V_{M,h}(\psi)$.

What I called thesis 1.

Proof: Take an arbitrary model M and variable assignments g and h. Take an arbitrary ψ. Clearly, if ψ has free variables, then the claim is (vacuously) true. So, assume that ψ has no free variables. Now, the claim that g and h agree on the free variables in ψ can be expressed like this (assuming that there are n occurrences of terms in ψ, and that we can list them $a_1, a_2,...,a_n$.): for all a_i in ψ, if a_i is an occurrence of a free variable, then $[a_i]_{M,g}=[a_i]_{M,h}$. So, in the case that ψ has no free variables, it is clear that g and h agree on the free variables in ψ since, for all a_i, the antecedent of the conditional fails to hold. Thus, by proposition 1, $V_{M,g}(\psi)= V_{M,h}(\psi)$. Since M, g, h and ψ were arbitrary, we have our result.

It's not difficult to get the result we really need here, but you do have to keep track of what the target is.

The inductive step should say, for the case where we're building a new formula of the shape $\forall x \chi$:

if (for all g and h that agree wrt free variables in χ, $[\![\chi]\!]_{Mg} = [\![\chi]\!]_{Mh}$),

then (for all g and h that agree wrt free variables in $\forall x \chi$, $[\![\forall x \chi]\!]_{Mg} = [\![\forall x \chi]\!]_{Mh}$).

61. Consider:

(a) $\Gamma, \phi \models \psi$
(b) $\Gamma \models \phi \supset \psi$

and:

(e) $\phi \dashv\models \psi$ (that is, $\phi \models \psi$ and $\psi \models \phi$)
(f) $\models \phi \subset\supset \psi$

Prove that (a) iff (b); and prove that (e) iff (f). You may find that an earlier homework already gives most of the solution to one of these: if so, you can of course cite your work for that earlier problem without repeating.

SAmple solutions to first part.

61. (a) Suppose $\Gamma, \phi \models \psi$. Since $\Gamma, \phi \models \psi$, all models that make Γ and ϕ true make ψ true too. So for all models in which Γ is true, $\neg\phi$ or ψ is true. So by the meaning of $\supset$, $\Gamma \models \phi \supset \psi$.

Suppose $\Gamma \models \phi \supset \psi$. By the definition of $\models$, all models in which Γ is true are models in which $\phi \supset \psi$ is true. By the definition of $\supset$, all models in which Γ is true are models in which $\neg\phi$ or ψ is true. Therefore, all the models that make Γ and ϕ true will make ψ true, because if Γ is true then $\neg\phi$ or ψ is true, and by supposition $\neg\phi$ is false. Therefore, by the definition of $\models$, $\Gamma, \phi \models \psi$.

Therefore, $\Gamma, \phi \models \psi$ iff $\Gamma \models \phi \supset \psi$.

Homework Exercise 61

```
1) Assume that $ \Gamma , \phi \models \psi$. Take an arbitrary model
$M$ and assignment $g$ such that $M$ and $g$ model every formula in
$\Gamma$, i.e. $V_{M,g} (\Gamma)=1$. Now, either $V_{M,g} (\phi)=0$ or
$V_{M,g} (\phi)=1$. If the former then $V_{M,g} (\phi \rightarrow
\psi)=1$. If the latter, then, by assumption, $V_{M,g} (\psi)=1$, and
so $V_{M,g} (\phi \rightarrow \psi)=1$. Thus in any event, $V_{M,g}
(\phi \rightarrow \psi)=1$. And since $M$ and $g$ were arbitrary, we
have our result.

Now suppose that $ \Gamma \models \phi \rightarrow \psi$. Take an
arbitrary model $M$ and assignment $g$ such that $V_{M,g} (\Gamma)=1$
and $V_{M,g} (\phi)=1$. Then, by assumption, $V_{M,g} (\phi
\rightarrow \psi)=1$. Suppose that $V_{M,g} (\psi)=0$. Then we have
that $V_{M,g} (\phi \rightarrow \psi)=0$. Contradiction! Thus,
$V_{M,g} (\psi)=0$. And since $M$ and $g$ were arbitrary, we have our
result.
```

typo, this should be 1.

64. How do you establish that a set of premises *doesn't* logically entail some result? Show that:

a. $Rab \not\models \exists x Rxx$
b. $Rab \not\models \neg\exists x Rxx$

Show that:

c. $\exists x \forall y Rxy \models \forall y \exists x Rxy$

but:

d. $\forall y \exists x Rxy \not\models \exists x \forall y Rxy$

64. (a) The 'older than' relation is a counterexample. If a is older than b, then Rab, but nothing is older than itself, so $\neg\exists x Rxx$.

(b) The 'has the same name as' relation is a counterexample. If a has the same name as b then Rab, but it is also the case that a has the same name as a, and so $\neg\neg\exists x Rxx$.

These 2 solutions give the right basic idea; there is some use/mention sloppiness here that you should avoid, but the core proposals are OK.

64.

a. Let R be such that for any x and y in the domain, either Rxy or Ryx, but never Rxx. Then Rab is true for some a and b, but this does not entail that $ \exists x Rxx$. (The "older than" relation is a counterexample here).

b. Let R be such that for any x and y in the domain, either Rxy or Ryx, and let there be some objects that bear the R relation to themselves. Then Rab is true for some a and b, but this does not entail that $\neg \exists x Rxx$. (The "cooks for" relation is a counterexample here).

The use/mention sloppiness in those examples is: are (a) and (b) constants in our object language or metalinguistic names/variables picking out objects in the domain? They seem to be playing both roles, and while that can be fine for, say, an email, you shouldn't be so loose in a more formal setting.

EXERCISE 64. We must exhibit a model $\mathcal{M}$ and some variable assignment g for $\mathcal{M}$ that yield a valuation on which every formula in the premise set takes the value 1 and the formula that is the conclusion takes the value 0.

a) $\mathcal{D} = \{u, v\}$
$\mathcal{I}(R) = \{\langle u, v\rangle\}$
$[a] = \mathcal{I}(a) = u$
$[b] = \mathcal{I}(b) = v$

Take an arbitrary variable assignment g. From $[a]_g = \mathcal{I}(a) = u$ and $[b]_g = \mathcal{I}(b) = v$, and $\mathcal{I}(R) = \{\langle u, v\rangle\}$, it follows that $\langle [a]_g, [b]_g\rangle \in \mathcal{I}(R)$, and hence $V_g(Rab) = 1$. However, there is no object $o \in \mathcal{D}$ such that $\langle [x]_{g^o_x}, [x]_{g^o_x}\rangle \in \mathcal{I}(R)$, and hence $V_g(\exists x Rxx) = 0$.

b) Take the above model except $\mathcal{I}(R) = \{\langle u, u\rangle, \langle u, v\rangle\}$. For reasons analogous to those given above, we get $V_g(Rab) = 1$. However, there is an object $o \in \mathcal{D}$ such that $\langle [x]_{g^o_x}, [x]_{g^o_x}\rangle \in \mathcal{I}(R)$, namely $[a] = \mathcal{I}(a) = u$, and hence $V_g(\exists x Rxx) = 1$, i.e. $V_g(\neg\exists x Rxx) = 0$.

Here is a more rigorous solution.