Homework Exercises due 10 November

Sample answers in purple.

94. Answer the following:

    a. If a relation is transitive and reflexive, must it be symmetric? No
    b. If a relation is transitive and irreflexive, must it be asymmetric? Yes
    c. If a relation is transitive and symmetric, must it be reflexive? Tricky! The answer is no, because the antecedent permits there to be objects that don't stand in the relation to anything, including themselves.
    d. If a relation is asymmetric, must it be irreflexive? Yes
    e. What's a more familiar name for a symmetric pre-order? An equivalence relation --- it's transitive, reflexive, and symmetric. An unintuitive limiting case of a kind of "order" relation.
    f. What is the difference between a weak and a strict partial order? Weak orders are reflexive, like $\leq$; strict orders are irreflexive, like $<$. For partial orders, the other requirements are transitivity and anti-symmetry. (Anti-symmetry plus irreflexivity, as in the case of a strict partial order, amounts to asymmetry.)
    g. What is the difference between a pre-order and a partial order? Partial orders add the requirement of anti-symmetry. So a and b can only stand in the relation symmetrically when $a = b$. Weak pre-orders permit distinct objects to stand in the relation symmetrically.
    h. What is the difference between a partial order and a total order? Total orders are partial orders where every pair of distinct objects are comparable. For a *strict* order, this means we'll have trichotomy: *exactly* one of aRb, $a = b$, or bRa will be true.
    i. What does it mean for a relation to be trichotomous? See end of previous answer. Many of you didn't notice that trichotomy requires the options to exclude each other: that is, that one *and only one* of them obtain.
    j. What does it mean for a relation to be Euclidean? If aRb and aRc then bRc (and cRb).
    k. What does it mean for a relation to be serial? $\forall x \exists y.\, xRy$
    l. What does it mean for a relation to be dense? Whenever xRz, there's some y (other than x and z) such that xRy and yRz. Among familiar numbers, the rationals and reals are dense, on their standard ordering.
    m. What does it mean for a and b to be comparable wrt some relation? One of aRb or bRa is true.
    n. What is the "ancestral" or transitive closure of the relation "is a parent of"? Might it be, perhaps, the relation of being an "ancestor"? Often when taking the transitive closure of a relation one also allows the identity relation as a limiting case, which may make a difference if the relation wasn't reflexive to start with. In that case, the ancestral of "parent" would be "is identical to or an ancestor of".
    o. What is the equivalence class of Ben under the relation "has the same height as"? the class/set of people (or whatever the relevant universe is) who have the same height as Ben, including Ben himself
    p. Is the relation "is at least as old as" symmetric, anti-symmetric, or asymmetric? Tricky! None of the above. Some of you chose anti-symmetric: but that would imply that you and I are at least as old as each other (that is, if we have the same age), we are numerically identical.

q. Is the relation "is a brother of" symmetric? Depends on what the relevant domain or universe is. Amongst boys, yes. Amongst siblings of mixed genders, no.
r. What is the difference between a maximal element, a greatest element, and a least upper bound (also called a "supremum") wrt some ordering relation? Maximal elements: nothing is R (e.g. longer than) it. There may be several of these; consider the maximally long words in the dictionary. Greatest element: it's R everything else in the relevant set. When this exists, it is unique. (There *could* be a longest word in the dictionary, for all I know.) The previous notions are defined for a specified ordering relation and set. The notion of an upper bound is defined against a richer background: our target set needs to be a subset of a (potentially) larger set on which the ordering relation is also defined. An upper bound of the order relation must be part of the larger set, and may or may not be part of the smaller set; it will be an object that everything (else) in the smaller set stands in order relation to. A least upper bound will be an upper bound which stands in the order relation to any other upper bound. When a least upper bound is part of the smaller set, it is a greatest element with respect to that set and order.
s. If an ordered set has any upper bounds, must it have a least upper bound? No. If the ordered set is dense, at least in the region where the smaller set stops and the larger set continues, there might be smaller and smaller upper bounds all of which are still outside the smaller set. Any interval of reals will have a least upper bound, on the standard ordering; but an interval of rationals need not. (This is the fundamental difference between rationals and reals.) To use an example one of you cited: let the smaller set be the subset of rationals less than $\sqrt{2}$. There is no largest rational in that set, and neither is there any smallest rational outside of that set. So the set has many upper bounds (any rational greater than $\sqrt{2}$); but no least upper bound. Another example: consider the set of strings $\{ ab, ba, aba, bab, baba \}$ ordered by the relation "is a substring of". The smaller set $\{ ab, ba \}$ has three upper bounds in the larger set, none of which belong to the smaller set and none of which is least.

95. What does it mean for an ordering relation on some set to be well-founded? Is the relation "is an initial substring of" on the set of strings over some alphabet well-founded? Is the set well-ordered by that relation? Well founded: any non-empty subset has some minimal element(s). Yes, that relation is well-founded on that set; but it might not be total, and so fail to be a well-ordering. If the strings are generated from a single letter, the relation will be total; but in general it won't be. Distinct strings can be such that neither is an initial substring of the other. When we have a well-founded relation that's total, and so a well-ordering, the minimal element in each non-empty subset will be, more strictly, a least element and so unique.

96. Try to explain each of the following in language that's less technical and which you're sure you understand, but is as rigorous as is compatible with the first requirements.

a. some theory is deductively consistent The theory doesn't prove any sentence and its negation. (There are multiple ways to define consistency.) If "theories" are understood to be deductively closed, this will entail that the theory doesn't *include* any sentence and its negation; but the former definition is more general as it doesn't make any assumptions about whether "theories" are understood so as to always be deductively closed.
b. some theory is deductively closed Any formula provable from premises in the theory is also in the theory.
c. some theory is deductively complete This assumes that we've specified what the theory's language is. It means that for every wff in the language (or perhaps we restrict ourselves to

sentences), either that formula or its negation is provable from the theory. As with the definition of consistent, this might be phrased in terms of being $\in$ the theory rather than in terms of being provable from the theory.

d. some theory is effectively decidable The question whether some arbitrary formula is provable from (or belongs to) the theory is answerable via some "effective" (mechanical/no-ingenuity-required) procedure, guaranteed to deliver the correct answer after a finite number of steps.

e. some theory is effectively enumerable Some "effective" procedure can list the formulas provable from (or belonging to) the theory, in such a way that for any correct answer, eventually (after some finite number of steps) that answer will be listed. (And no incorrect answer is ever listed.) Generally it is not prohibited that the same answer be listed multiple times; but that can be prohibited if you like.