Chapter 1
Intensional Logic and Semantics for Deontic Modals

> *"What definition can we give of 'ought', 'right', 'duty', and other terms expressing the same fundamental notion? To this I should answer that the notion which these terms have in common is too elementary to admit of any formal definition. [...] I find that the notion we have been examining, as it now exists in our thought, cannot be resolved into any more simple notions: it can only be made clearer by determining as precisely as possible its relation to other notions with which it is connected in ordinary thought, especially to those with which it is liable to be confounded."*
>
> (Sidgwick 1874/1981, pp. 32–33)

1.1 Introduction

This chapter is a critical introduction to different (classical and non-classical) semantics for deontic modals, in particular semantics given to 'ought'.

After some methodological considerations on the role of semantic and philosophical inquiry on modality and natural language modality (Sect. 1.1), I briefly review what is now considered the classical semantic proposal for (deontic) modals, as given mainly by Angelika Kratzer from the 70s onwards (Sect. 1.2), which under minor assumptions is proved equivalent to the possible world semantics for standard deontic logic. In particular, I consider extensions of the standard semantic proposal to account for features such as strong versus weak necessary modals.

In Sect. 1.3 I expound on some issues considered to be problems for the classical semantics (namely: monotonicity, normative dilemmas, information sensitivity, deontic conditionals, gradability, and scope order), and see to what extent both classical and non-classical semantics can cope with them.

In this chapter I aim to suggest that (at least some of) the problems of the (classical and non-classical) semantics for deontic modals have to do with their being essentially (albeit sometimes covertly) extensional, or too coarse-grained. I will later introduce some finer-grained tools that could be used to tackle these problematic issues.

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On the other hand, due to space limitations and how the rest of this work develops, I will set aside, for the most part, imperative-based semantics, primitive dyadic operators, and the issue of different primitive permissions.

Some Methodological Considerations on Modality and Natural Languages Philosophers studying modality have a tough job ahead of them. There is in fact a constant tension between natural language data, and philosophical intuitions. On the one hand, modality in natural languages is both synchronically and diachronically a very complex phenomenon. Synchronically, even conceiving modality narrowly, most languages do not have a closed class of modals elements, i.e. of modal verbs. Diachronically, modal elements acquire non-modal meanings or uses, and conversely non-modal elements acquire modal meaning or use, losing the original, pre-modal, ones (so-called desemanticization or bleaching)[1] or not.[2]

Even setting aside the philosophical questions about what modality is, it is still hard to come up with convincing acceptable criteria to isolate a closed class of modal elements. The first intuitive macro distinction would be between modal *verbs* and other modal words. But even among verbs, which ones are *modal* verbs? The easier thing to do would be to identify morphological criteria. Modal verbs may have (i) a unique, specific morphology (as in most Germanic languages); (ii) a defective morphology (English, Irish Gaelic, some Baltic and Albanian forms); (iii) an impersonal morphology (a nice crosslinguistic example is English 'maybe', 'may be' and Polish 'może' (for a survey, cf. Hansen and de Haan 2009a, pp. 521ff.).

Morphosyntactic criteria are, however, language-specific. The move to semantics, then, seems the natural next step.

Natural language data push for *disunity*: disunity across languages (some languages do not have strong obligation modals), disunity across time (a deontic modal becomes epistemic, or gradually disappears), and intralinguistic disunity (modality is realized not only with auxiliary, but with adverbs, adjectives, or regular verbs, specialized in meaning or not). Philosophical and logical theories, on the other hand, seek unity, abstracting away from "contingent issues". Just three examples: *First*, although there has been some attention recently on so-called "weak" necessity modals, like 'ought' and 'should', with regard to "strong" necessity modals, like 'have to' and 'must', there is no discussion (as far as I am aware), in the contemporary literature,

[1] One example of bleaching is the English 'may', which lost its original meaning of having strength. One example of the acquisition of modal meaning without losing the pre-modal one is the Italian form 'avere (da)', to have, also to have to.

[2] Some patterns of modal meaning acquisition are well established. Here are some (cf. Hansen and de Haan 2009a, pp. 520–1):

- want > need > must (Arabic, Balto-Finnic, Hungarian, Romance);
- strength > power > possibility (Albanian, Baltic, Germanic, Slavonic);
- get > possibility (Balto-Finnic, Icelandic, Irish);
- know > can > possibility (Arabic, Hungarian);
- possession > must (Arabic, Baltic, Hungarian, Germanic, Romance, Slavonic).

The linguistic literature on these topics is extensive. For an interesting survey of natural language modality, albeit limited to the languages of Europe, see Hansen and de Haan (2009b).

between 'may' and 'might', which, at least historically and even nowadays in some corpora, display the strong-weak pattern of must-ought. On the one hand, 'may' is losing its deontic flavor, becoming eminently epistemic. On the other hand, most native speakers no longer hear the difference in tense, if not in strength, between 'may' and 'might', yet it is recognized in descriptive linguistics, in corpora, and dictionaries.

Second, take 'will'. As a primitive sense, it is volitional. It is now mostly used as an auxiliary for the future. 'Shall', instead, had a deontic meaning (cf. German 'sollen', and English 'should'), but is now rarely used as a polite form in questions and as a future auxiliary in the first person. Yet, those primitive senses (volitional for 'will', deontic for 'shall') have not disappeared completely: just compare:

(3) "Shall I call you a car?"

(4) *"Will I call you a car?"

(5) "Will you call her a car?"

Third, more and more frequently (at least in British English, considering the corpora) strong obligation is not expressed with 'must' or 'ought', but rather with 'need to', or 'supposed to'. Yet, there is no theory of 'need to' and 'supposed to' that I know of.[3]

What is the contemporary philosopher, or the philosophically inclined linguist, supposed to do, even with these borderline, yet not insignificant, cases? The descriptive linguist may afford to analyze the use and semantics of each modal construction, be it adverbial, auxiliary, or else. The philosopher, however, needs to find a *via media* between useless proliferation and disunity, and extreme oversemplification.

Rather than describing natural language phenomena, I will try to work with a general approach which aims for a regimentation of natural language phenomena.

It is also for these reasons that I start my work analyzing the deontic modal *par excellence*: 'ought', even though it is not used so frequently anymore, and has fresher and stronger competitors. Non-deontic readings are set aside.

1.2 Kripke-Kratzer Logic and Semantics for Deontic Modals

1.2.1 The Basics

In this brief section, I first lay out the basic tenets of predicate modal logic in an informal way, and its application to formal semantics as found in the mainstream literature.

[3]This phenomenon is not limited to English, of course. Cf., for deontic necessity, the Italian 'bisogna', 'conviene', 'occorre'.

Logical syntax defines the language and well-formed formulas; primitive elements of our language are:

1. a set of individual constants $a, b, c, \ldots$ which may be either finite or countably infinite;
2. a countably infinite set of individual variables $x_0, x_1, \ldots$;
3. a set of predicates of finite arity P_i^n;
4. logical connectives: $\neg, \wedge, \vee, \rightarrow$;
5. quantifiers: $\forall, \exists$;
6. (,);
7. for modal logics, the operators $\Box, \diamond$.

The semantics specifies a model $\mathcal{M}$ made of a set D of individuals, a function F specifying the interpretation of non-logical constants, i.e. a function mapping each individual constant to a member of D, and each n-ary predicate P to a set of n-ples of individuals. We need moreover a function g assigning an interpretation to individual variables. This function must be distinguished from F because variables may change from one world to another.

The denotation of an expression ϕ, relative to a model $\mathcal{M}$ and to an assignment g ($[\![\phi]\!]^{\mathcal{M},g}$) is recursively defined as expected:

1. If ϕ is a non-logical constant, then $[\![\phi]\!]^{\mathcal{M},g} = F(\phi)$
2. If ϕ is a variable, then $[\![\phi]\!]^{\mathcal{M},g} = g(\phi)$
3. If P n-ary predicate and $t_1, \ldots, t_n$ are terms, then $[\![P(t_1, \ldots, t_n)]\!]^{\mathcal{M},g} = 1$ iff $([\![t_1]\!]^{\mathcal{M},g}, \ldots, [\![t_n]\!]^{\mathcal{M},g}) \in [\![P]\!]^{\mathcal{M},g}$
4. $[\![\neg\phi]\!]^{\mathcal{M},g} = 1$ iff $[\![\phi]\!]^{\mathcal{M},g} = 0$
5. etc.
6. $[\![\forall x\phi]\!]^{\mathcal{M},g} = 1$ iff for all $d \in D$, $[\![\phi]\!]^{\mathcal{M},g^{d/x}} = 1$
7. $[\![\exists x\phi]\!]^{\mathcal{M},g} = 1$ iff there is at least one $d \in D$ such that $[\![\phi]\!]^{\mathcal{M},g^{d/x}} = 1$

So far, semantical values are relativized to models and assignments. A formula is true in a model iff it is true in every assignment of that model.

Modal logic deals with such notions as necessity and possibility. Syntactically, modal logic is just an extension of propositional logic with one propositional operator (or two), for instance '$\Box$' for necessity (and '$\diamond$' for possibility), such that if A is a well-formed formula, $\Box A$ ($\diamond A$) is a well-formed formula.

When are $\Box A$ and $\diamond A$ true, i.e. what is their semantics? An extensional approach is not enough, and that is why there has been what we may call an "intensional revolution".

Valuations are now relativized to given worlds (to stay metaphysically neutral, one can think of worlds as indices whose meaning varies) such that we now have an intensional model: $\mathcal{M}^{\mathcal{I}} = < \mathbb{W}, D, \mathcal{R}, I, g >$, where $\mathbb{W}$ is the set of worlds, D is the domain of individuals, $\mathcal{R}$ is a binary relation on W, I specifies which individuals are at which world and g assigns values to variables.[4]

[4] As everyone knows, relational semantics is (i) not the only semantics available for modal logic; (ii) only moderately useful. Algebraic, topological, measure-theoretic and category-theoretic semantics

All semantic values of extensional contexts are as before (but in a given $w \in W$), but we can now define specific semantic clauses for intensional operators:

1. $[\![\Box\phi]\!]^{\mathcal{M},g,w} = 1$ iff for all w' such that $w\mathcal{R}w'$, $[\![\phi]\!]^{\mathcal{M},g,w'} = 1$
2. $[\![\Diamond\phi]\!]^{\mathcal{M},g,w} = 1$ iff there is at least one w' such that $w\mathcal{R}w'$, $[\![\phi]\!]^{\mathcal{M},g,w'} = 1$

Propositions are taken to be sets of (possible) worlds, which are themselves further unanalyzed particulars.[5]

Equivalently, propositions may be considered as functions from worlds to truth-values, i.e. the characteristic functions of those sets, as long as we have only two truth-values, i.e. true and false, and associate them to 1 and 0 respectively.

This semantics can also be seen set-theoretically: semantic value operations are explained by set operations (in the following, $[\![A]\!]$ is the set of worlds where the proposition A is true):

1. $[\![\neg A]\!] = W - [\![A]\!]$.
2. $[\![A \wedge B]\!] = [\![A]\!] \cap [\![B]\!]$.
3. $[\![A \vee B]\!] = [\![A]\!] \cup [\![B]\!]$.
4. $[\![A \rightarrow B]\!] = (W - [\![A]\!]) \cup [\![B]\!]$.

Set-theoretic and algebraic tools make formal semantics a very powerful and elegant theory. But they also make it prone to suffer from the same limitations set theory has to model a very fine-grained, non-mathematical context, as will become clear below.

Several proposals to extend this semantics to deontic modalities have been advanced. What follows has no pretense of historical accuracy, but rather of being a useful snapshot.[6]

Intuitively, a deontic accessibility relation R, according to a particular system of norms, rules, or standard, holds between w and w' iff all the rules the system establishes in w are followed in w'.[7] If in w, according to the normative system, theft is forbidden, then wRw' iff there is no theft whatsoever in w'.

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are also available and sometimes more perspicuous. The literature is vast, but the interested reader is referred to solid introductions like Chagrov and Zakharyashchev (1997) and Blackburn et al. (2002).

[5] Of course, the other way round, i.e. taking propositions as primitive, and defining (possible) worlds in terms of maximal consistent sets of propositions, is also possible. This venerable idea, based on the writings of Bolzano, Wittgenstein and C.I. Lewis, and which allows very elegant algebraic manipulations, is currently a minority view.

[6] For the record, it must be noted that the semi-official birth of contemporary deontic logic is identified with the works of von Wright and Kalinowski, none of whom adopted this kind of semantics. In von Wright's seminal paper (VonWright 1951), for instance, deontic operators range over expressions for action-types, making iterations or mixed sentences ill-formed. For a historical survey see Hilpinen and McNamara (2013). Of course there is also a prehistory of deontic logic, whose standard reference are Rosetus, Leibniz, Mally, among others.

[7] There are very many subtle issues in the vicinity of the rule-following problem. I set them aside for the time being.

1.2.2 Standard Deontic Logic

Standard Deontic Logic (hereafter SDL) is just the smallest normal modal logic K with the addition of scheme D: $\mathcal{O}p \rightarrow \neg\mathcal{O}\neg p$.[8] Semantically, D corresponds to the seriality of $\mathcal{R}$.

One possible axiomatization is the following[9]:

1. All propositional tautologies
2. $\mathcal{O}(p \rightarrow q) \rightarrow (\mathcal{O}p \rightarrow \mathcal{O}q)$ (K)
3. $\mathcal{O}p \rightarrow \neg\mathcal{O}\neg p$ (D)
4. If p, $p \rightarrow q$, then q (MP)
5. If $\vdash p$, then $\mathcal{O}p$ (NEC)

The only "modal" schemata or rules are K, common to all normal modal logics, the rule of Necessitation, which—at least in the deontic case—is quite contentious, as we will see shortly, and the deontic schema D, which guarantees a deontic consistency of sort. Obviously, one could take other modal rules and/or schemata as primitive; the result is a normal modal logic, slightly stronger than K, whose relational (Kripke-style, possible-world) semantic is straightforward.

A model $\mathcal{M}$ is defined as above. For the propositional case we have a domain of words (points, indices) W, a binary accessibility relation $\mathcal{R} \subset W^2$, such that if $w, w' \in W$, $w\mathcal{R}w'$ iff everything obligatory in w holds in w'.

A formula is valid iff true at every world in every model. Logical consequence is consequently defined model-theoretically.

The only constraint on the accessibility relation is seriality: for every world w, there is at least a world w', such that $w\mathcal{R}w'$, i.e. a world realizing what is obligatory in the former world.

The truth-functional connectives behave standardly. The semantic clauses for obligation and permission are analog to those for necessity and possibility in alethic modal logic[10]:

1. $[\![\mathcal{O}\phi]\!]^{\mathcal{M},w} = 1$ iff for all w' such that $w\mathcal{R}w'$, $[\![\phi]\!]^{\mathcal{M},w'} = 1$
2. $[\![\mathcal{P}\phi]\!]^{\mathcal{M},w} = 1$ iff there is at least one w' such that $w\mathcal{R}w'$, $[\![\phi]\!]^{\mathcal{M},w'} = 1$.

SDL is sound and complete with respect to Kripke-semantics with serial frames.

There is a provably equivalent semantics to the standard relational semantics given to KD (SDL), a semantics which makes use of an ordering (one for each world w), such that $w' \leq_w w''$ iff w'' is at least as good as w' with regard to w. This relation is usually taken to be a total pre-ordering (i.e. reflexive, transitive, and connected) and

[8] We present SDL not because it is the standard system of deontic logic, but rather as a standard of comparison.

[9] For other axiomatizations and an historical perspective see Hilpinen and McNamara (2013), McNamara (2014), and also Chap. 4.

[10] In a classical setting, (weak) permission is just the dual of obligation ($\neg\mathcal{O}\neg$). Similarly, from one notion taken as a primitive, say, obligation, one could define forbiddance ($\mathcal{O}\neg$), omissibility ($\neg\mathcal{O}$), optionality ($\neg\mathcal{O} \wedge \neg\mathcal{O}\neg$). Of course one could start with a different primitive. These definitions have obviously been questioned in the literature.

adding or not the Limit Assumption (more on that shortly) will make the relational semantics easy to derive.[11]

In particular, if we formulate the limit assumption as follows:

(6) for each w there's a w' s.t. for all w'', $w' \geq_w w''$

we could then formulate the clause for obligation as follows:

(7) $[\![\mathcal{O}\phi]\!]^{\mathcal{M},w} = 1$ iff $[\![\phi]\!]^{\mathcal{M},w'} = 1$ in all the best worlds w'.

Two considerations are in order. *First*, if we make the limit assumption, we can recover seriality. *Second*, there are at least two non-equivalent notions of 'best' we could mean: maximality and optimality. Maximality (a world is maximal if it is at least as good as all the others) is weaker than optimality (a world is optimal if there is no better world), but the two coincide if the ordering is connected. This is not a problem in the present setting, but will be for the literature following Kratzer's approach.

Back to the relational semantics. Seriality on $\mathcal{R}$, expressed syntactically by the D schema, plus irreflexivity, guarantees at least two worlds in the model. Does seriality require an infinite domain of worlds? It depends on other conditions imposed on $\mathcal{R}$. If we want $\mathcal{R}$ to be irreflexive, then it depends whether $\mathcal{R}$ is transitive or not. If it is transitive, then the relation cannot be symmetrical, and the domain must be infinite.[12]

But the relation required in KD (SDL) has to be only serial, and not transitive. Therefore, if not necessarily transitive, the model can be cyclical, i.e. the last world can be connected to the first and the model can be finite.

Of course, this may cause some philosophical perplexities of the following sort: The relation of deontic accessibility is taken to mean that if $w'\mathcal{R}w''$, then w'' realizes all the obligations holding in w', so w'' is somewhat better than w'. Suppose $w''\mathcal{R}w'''$. Then w''' realizes all the obligations holding in w'', being somewhat better than w''. In general, suppose that if $w^{n-1}\mathcal{R}w^n$, then w^n is somewhat better that w^{n-1}. If the model is cyclical, then at some point, $w^n\mathcal{R}w'$, meaning that w' realizes the obligations holding at w^n, being somewhat better than w^n.

If two conditions (a cyclical model, and the accessibility relation interpreted as betterness) were to hold, there would be an unacceptable result, namely, w' would be better than itself. Of course, what is wrong is to interpret $\mathcal{R}$ as a betterness relation, which is surely transitive, and, given irreflexivity and seriality, would need an infinite model. $\mathcal{R}$ is surely irreflexive but non-transitive, and therefore cannot be interpreted as betterness.

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[11] The equivalence of this ordering relation semantics to relational semantics for SDL has been proved in Goble (2003). For the use of ordering relations in the semantics of modality, see infra. (The idea of using preference orderings in deontic logic has been developed for instance in Hansson 1990, 2000.)

[12] *Proof* (*sketch*) Take an irreflexive transitive relation and require it is serial. Start from a one-world model. Another world is needed to fulfill the requirements on $\mathcal{R}$, since $\mathcal{R}$ must be irreflexive. Two worlds are not sufficient, because $\mathcal{R}$ cannot be symmetrical, and must be transitive. Therefore, to be serial, one needs to continue adding worlds to the model. Hence an infinite number of worlds is required.

1.2.3 Kratzer's Approach

Krazter's approach is now standard as a semantic theory of modals. There are two landmark ideas: the idea of relative modality (Kratzer 1977), and the idea of ordering semantics (Kratzer 1981).[13] *First*, modals such as 'must' or 'can' are not *ambiguous* between (say) a deontic and and epistemic reading; rather, they are *relative* to one or more sets of background assumptions, which Kratzer calls "conversational backgrounds" (Sect. 1.2.3.1). In this respect in Kratzer's view modal verbs are not semantically ambiguous (i.e. 'must' does not have multiple meanings), but rather what we may call pragmatically ambivalent.[14] *Second*, worlds are not considered simply in terms of their accessibility, but are ranked (partially ordered) *via* specialized conversational backgrounds, called ordering sources (Sect. 1.2.3.2).

Contextual and pragmatical issues do much more work in Kratzer's framework than supplying the spatiotemporal coordinates of the utterance and the identity of the speaker: they *de facto* select the kind of accessibility relation. 'Should', for instance, is not semantically ambiguous, but is (pragmatically) ambivalent: it is context that supplies the meaning of should in a particular utterance, and, formally, selects the accessibility relation between worlds.[15]

1.2.3.1 Conversational Background

Conversational backgrounds in Kratzer's linguistic analysis of modals *de facto* take the place of the accessibility relations in standard modal logic. To focus on the role of the conversational background, the implicit information supplied by context can be made explicit by a paraphrasis such as "in view of (the system of rules $\mathcal{S}$, what I know, …)".

Definition 1 A *conversational background* is a function mapping possible worlds to sets of propositions (cf. Kratzer 2012b, p. 31), i.e. to sets of sets of worlds.

Given Kratzer's assumptions about the fact that modals are not semantically ambiguous, introducing conversational background makes sense: whether a modal verb receive an epistemic, deontic, doxastic, …, interpretation depended technically on the kind of accessibility relation, and now, it depends on the conversational background. It is a higher order explanation: the domain of quantification is selected by context, but in a technically precise way. Given a conversational background f, for any worlds w and w', wRw' iff every proposition in $f(w)$ is true in w'. If f is deontic, $f(w)$ stands for the set of rules, and wRw' iff the set of rules $f(w)$ are followed

[13] The original papers were written in the '70s and '80s, but have now been revised and are published in Kratzer (2012a). I will consider the newest version and will quote them respectively as Kratzer (2012b, c).

[14] On the distinction between semantic ambiguity and pragmatical ambivalency, cf. Conte (2007).

[15] For an introductory exposition, cf. Portner (2009), p. 51ff.

in w'. The set of worlds accessible from w is therefore $\bigcap f(w)$. (In the following we generally assume $\bigcap f(w) \neq \varnothing$, unless otherwise noted.)

1.2.3.2 Ordering Source

Imagine S. ought to be punished (for he engages in underage prostitution, let's say, which is illegal). 'Ought' is deontic, so we treat it like a universal quantifier over the worlds mapped to our worlds, and in particular all the worlds realizing the laws of our worlds, i.e. all the worlds where S. is in fact punished.

But the worlds where S. is fact punished are in fact worlds where S. engaged in underage prostitution. So, the proposition that S. engages in underage prostitution is true in those worlds, which are all those accessible from our world. Therefore, it follows, according to the simple view above, that S. ought (deontically) to engage in underage prostitution, which is surely a counterintuitive, albeit not impossible, result.

This problem is caused by the fact that in our mock example the conversational background conflated factual information and the content of the law.

In her 1981 paper, Kratzer holds that a modal is to be interpreted with reference to two conversational backgrounds, a *modal base* f and an *ordering source* g, which we add to our valuations as indices. If $f(w)$ is a conversational background, $\leq_{g(w)}$ is an ordering generated by the set of propositions $g(w)$, the ordering source.

The modal base provides the set of relevant worlds, whereas the ordering source obviously orders the worlds as $\leq_{g(w)}$: both are functions from worlds to sets of propositions.

In our example, the modal base selects the worlds where in fact S. engages in underage prostitution, and only then these worlds are ranked according to our laws.

A deontic necessity modal, then, would be interpreted as universal quantifier over all the best worlds accessible from ours:

(8) $[\![\text{must}\phi]\!]^{w,f,g} = 1$ iff for all $u \in \bigcap f(w)$, there is a $v \in \bigcap f(w)$ such that $v \leq_{g(w)} u$ and for all $z \in \bigcap f(w)$, if $z \leq_{g(w)} v$, then $z \in \phi$.

Let $L = \{a, b, c\}$ be the set of our laws, expressed as simple "descriptive" propositions. The best worlds, let's say, are those realizing more propositions:

$\{a\}, \{b\}, \{c\} \leq \{ab\}, \{ac\}, \{bc\} \leq \{abc\}$

Clearly the ordering is not total: there are ties, i.e. worlds realizing one law each, or two laws each, cannot be compared.[16]

But in this toy model, deontic necessity modals will be checked against the best worlds, i.e. those realizing $\{abc\}$.

There are some difficulties with orderings: *first*, it may be that two propositions are inconsistent, and therefore there are two different (incompatible) sets of "best"

[16] A total ordering is a total, anti-symmetric, transitive relation among all members of the domain: $\leq$ on $\mathbb{N}$ is a such a total ordering. A partial ordering is a reflexive, anti-symmetric, transitive relation among some members of the domain: not every two elements need to be related. A pre-order lacks anti-symmetry.

worlds, as when there are two contradictory laws in our starting set L, i.e. laws that cannot be obeyed at the same time. *Second*, there may be an infinite number of propositions such that there is no best world, but one can always find a better world.

There are obvious workarounds. We can make the so called *limit assumption*, i.e. we can assume there is always at least a best world.[17]

The limit assumption is usually made in linguistics, both for practical reasons and to have a uniform paradigm with regard both to modal logic and other kinds of quantification.

In any case, Kratzer (1981), the classic text for ordering semantics for deontic modals, does not make the limit assumption.

With the limit assumption, one can say that for a modal sentence (with only one modal verb) the accessible worlds are the best-ranked (according to $\leq_{g(w)}$) relevant worlds (in $\bigcap f(w)$). However, since the ordering is not total, one needs to specify whether "best" is understood as optimality or maximality and adapt the clause in the obvious way.

As far as the logic is concerned, Kripke semantics and Kratzer semantics are equivalent, as shown by the following

Theorem 1 *If $\bigcap f(w) \neq \varnothing$, then SDL is sound and complete with respect to Kratzer's semantics.*[18]

Proof Cf. Theorems 1, 3 in Goble (2013b). □

Theorem 2 *For every Kripke serial model there is an equivalent Kratzer model (with some form or other of limit assumption).*

Proof Cf. Theorem 2 in Goble (2013b).[19] □

Without the limit assumption, things get a bit messier. Since this is just an introductory survey, I can only gesture, but not discuss Swanson's recent paper on partial preorders and the limit assumption (cf. Swanson 2011), to which I refer the curious reader.

A Philosophical Problem Although introduced as a technical move to solve some problems, the ordering builds a substantial philosophical thesis in the semantics: (at least the meaning of) deontic modality is reduced to the good, or to value.

While this is a perfectly acceptable metanormative position, and has been argued for from several points of view, the reduction of the deontic to the axiological, so to

[17] On whether we should make the limit assumption, particularly in the theory of counterfactual conditionals, see Lewis (1973), who is against the limit assumption, and Stalnaker (1987), who is in favor, also because a situation with an infinite number of options to rank would never occur practically. Of course the extension of the applicability of the limit assumption in the theory of counterfactuals to general ordering semantics is not without issues, as Portner (2009), p. 66, notes.

[18] One should adapt the nonemptiness conditions to maximal and optimal worlds respectively.

[19] I obtained these results with a detour to neighborhood semantics first, before I discovered Goble's proof, which was obtained earlier than mine and is neater.

speak, cannot be simply a technical move, but needs a philosophical justification. (A possible reply would be: well, formal semantics uses possible worlds as a technicality, without being committed or even interested in ontological disputes about their nature. True, but the analogy cannot be made: the task of formal semantics is not to give a convincing metaphysical picture, but rather (in this particular case), to explain the meaning of (deontic) modality, a task to which reduction of the deontic to the axiological is much more relevant.)

1.2.4 Scott-Montague Semantics

Whereas the Kripke-Kratzer semantics is rather successful on a number of counts, there are several problems highlighted in the literature. This fact gave rise to a lively debate between those pushing for radical change and those proposing amendments and modifications to the classical paradigm. The next section introduces some of those problems and new proposals.

However, already in the logic tradition, Dana Scott and Richard Montague independently discovered neighborhood semantics, a generalization of Kripke relational semantics aimed at covering non-normal logics too. A neighborhood frame (W, N) has a neighborhood function (instead of an accessibility relation) $N : W \to 2^{2^W}$ which associates (intuitively) to each world the propositions necessary in that world, such that the standard clause for necessity becomes:

$$\mathcal{M}, w \models \Box\phi \text{ iff } |\phi|^{\mathcal{M}} \in N(w)$$

where $|\phi|^{\mathcal{M}}$ is the truth set of $|\phi|$.

In what follows I will not be concerned with neighborhood semantics due the following fact:

Fact 1 *The law of substitution of logical equivalents (SLE) is valid in the class of all neighborhood frames.*

Proof Suppose C is a class of neighborhood models such that $\vDash_C A \leftrightarrow B$. Then, the truth set of A is just the truth set of B, in all $\mathcal{M}$. Therefore for all worlds in any model, the truth set of $A \in N(w)$ iff the truth set of $B \in N(w)$, which is just to say that for any w in any model, $\mathcal{M}, w \vDash \Box A$ iff $\mathcal{M}, w \vDash \Box B$, that is, $\vDash_C \Box A \leftrightarrow \Box B$. □

This means that neighborhood semantics cannot characterize logics weaker than E, i.e. where the substitution of classical logical equivalents would fail.

1.3 Problems and Paradoxes for Classical Semantics and SDL

In this section I report some problems and paradoxes which have been discussed in the literature. First I present some classical paradoxes arising from the literature on SDL. Then I present some problems arising from the formal semantics literature. This division is out of mere presentational convenience, since SDL is sound and complete with regard to the version of Kratzer's semantics we have seen.

1.3.1 Problems and Paradoxes in the Logic Literature

We start by reviewing simple versions of the most well-known paradoxes of SDL. The literature on these puzzles and paradoxes is vast. For a survey, with plenty of references to the original literature and some discussion, see Hilpinen and McNamara (2013).

1.3.1.1 Free Choice Permission

The free choice permission puzzle is as follows:

(9) You may fence left-handed or fence right-handed.

(10) You may fence left-handed and you may fence right-handed.

It is plausible that (10) is entailed by (9): if I have the permission to fence left-handed or right-handed, it seems to follow that I have the permission to fence in the first way and the permission to fence in the second way, barring specific or contingent difficulties (until some time ago you needed to be wired to your épée, and switching hand during a duel could have been complicated, if not forbidden or discourteous to your opponent).

However, given a plausible formalization of (9) and (10) respectively as

(11) $\mathcal{P}(l \vee r)$

(12) $\mathcal{P}l \wedge \mathcal{P}r$

there is no derivation of (12) from (11) as an instance of **FC**:

$$\mathcal{P}(\phi \vee \psi) \rightarrow \mathcal{P}\phi \wedge \mathcal{P}\psi \tag{1.1}$$

But assume there were a theorem such as **FC**.

Then by SLE and the definition of permission, it would follow that everything is permitted, or nothing obligatory.

1.3.1.2 Ross's Paradox

(13) You ought to mail the letter.

(14) You ought to mail the letter or burn it.

Plausible formalizations in SDL:

(15) $\mathcal{O}m$

(16) $\mathcal{O}(m \vee b)$

In SDL, If $\mathcal{O}m$ then $\mathcal{O}(m \vee b)$.[20]

But 16 does not follow from 15: if you burnt the letter, you would fulfill an obligation you have, according to SDL. This is implausible: there is no obligation you have that could be fulfilled by burning the letter.[21]

1.3.1.3 The Good Samaritan Paradox

The Good Samaritan Paradox was originally proposed by Prior (1958), with a version of necessitation using forbiddance rather than obligatoriness. Here is an equivalent formulation:

(17) It ought to be the case that Jones helps Smith who has been robbed.

(18) It ought to be the case that Smith has been robbed.

there is no plausible derivation of (18) from (17), even and under the further (plausible) assumption that Jones helps Smith who has been robbed if and only if Jones helps Smith and Smith has been robbed. Regardless of Prior's concrete examples, many cases, such as Professor Procastinate, or Nurse, display a logical structure whose plausible formalization is:

(19) $\mathcal{O}(h \wedge r)$

(20) $\mathcal{O}r$

but

$\mathcal{O}(h \wedge r) \rightarrow \mathcal{O}r$[22]

according to which (roughly) if it ought to be the case that Jones helps Smith who has been robbed then it ought to be the case that Smith has been robbed.

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[20] *Proof* From $\vdash p \rightarrow (p \vee q)$, apply deontic necessitation: if $\vdash p \rightarrow (p \vee q)$ then $\vDash \mathcal{O}p \rightarrow \mathcal{O}(p \vee q)$.

[21] Recent bimodal approaches to Ross's paradox, i.e. using a (normal) metaphysical or alethic modality beside 'ought' in order to restrict some deontic rule or other to compossible situations only (see for instance Danielsson 2005, 2007) have been convincingly defused, in the case that 'ought' remains congruential (i.e. non-hyperintensional) by Humberstone (2016), pp. 324–330.

[22] *Proof* $\vDash h \wedge r \rightarrow r$ by propositional classical logic. Apply necessitation.

1.3.1.4 The Gentle Murder Paradox

(21) Oedipus ought not to kill his father.

(22) If Oedipus does kill his father, then Oedipus ought to kill him gently.

(23) Oedipus kills his father.

Here is a possible formalization in SDL:

(24) $\mathcal{O}\neg k$

(25) $k \rightarrow \mathcal{O}g$

(26) k

By *modus ponens*, from (25) and (26), it follows that $\mathcal{O}g$.
By adding the following claim

> Oedipus kills his father gently only if Oedipus kills his father.

symbolized as

(27) $g \rightarrow k$,

as a logical truth in an expanded system, by Inheritance we have

(28) $\mathcal{O}g \rightarrow \mathcal{O}k$,

and by *modus ponens*,

(29) $\mathcal{O}k$.

From the fact that if Oedipus kills his father he ought to kill him gently, and from the fact Oedipus kills his father, the conclusion we obtain in SDL (that Oedipus ought to kill his father) is not intuitively acceptable.

By the alleged duality of obligation and permission, from (29) we can add

(30) $\neg O\neg k$,

which, together with (24), gives us a contradiction.

Assuming this is an acceptable formalization, there are at least two ways to avoid the conclusion that Oedipus ought to kill his father. Either we drop *modus ponens* or we drop Inheritance or Necessitation, for instance by refusing the substitutivity of logical equivalents.

1.3.1.5 The Miners' Paradox

> Ten miners are trapped either in shaft A or in shaft B, but we do not know which. Flood waters threaten to flood the shafts. We have enough sandbags to block one shaft, but not both. If we block one shaft, all the water will go into the other shaft, killing any miners inside it. If we block neither shaft, both shafts will fill halfway with water, and just one miner, the lowest in the shaft, will be killed (Kolodny and MacFarlane 2010).

Thus it seems plausible that:

(31) You ought to block neither shaft.

Using a plausible formalization like

(32) $inA \vee inB$

(33) $inA \rightarrow \mathcal{O}(blockA)$

(34) $inB \rightarrow \mathcal{O}(blockB)$

one derives

(35) $\mathcal{O}(blockA) \vee \mathcal{O}(blockB)$

which contradicts the assumed "right" conclusion (31).

According to Kolodny and MacFarlane (2010), one could block the paradox in four ways: by rejecting one or more premises, by distinguishing objective and subjective senses of 'ought', by taking the argument to have a "non-obvious" logical form (wide scope modality or dyadic obligation), or by taking the argument to be invalid even in its obvious logical form. In this latter case, one needs to deviate from classical logic. Another way is to be fine with admitting inconsistent obligations.

Kolodny and MacFarlane (2010) reject *modus ponens*. A discussion of various solutions to this paradox is postponed until Chap. 4.

1.3.1.6 Contrary-to-Duty

Here is the informal set-up:

(36) It ought to be that Jones goes to assist his neighbors.

(37) It ought to be that if Jones goes, then he tells them he is coming.

(38) If Jones does not go, then he ought not tell them he is coming.

(39) Jones does not go.

And the usual formalization (taken from McNamara 2014):

(40) $\mathcal{O}g$

(41) $\mathcal{O}(g \rightarrow t)$

(42) $\neg g \rightarrow \mathcal{O}\neg t$.

(43) $\neg g$

Here I will not consider other ways of expressing (36)—(39), or introducing a dyadic operator. Here is the usual way to derive an inconsistency from (40)—(43):

(44) $\mathcal{O}t$ from (40), (41), K and MP

(45) $\mathcal{O}\neg t$ from (42), (43), MP

(46) $(\mathcal{O}t \wedge \mathcal{O}\neg t) \rightarrow (\mathcal{O}(t \wedge \neg t)$ agglomeration

(47) $\mathcal{O}q$ PC, deontic explosion

Here is the argument for explosion in PC.

1. $\alpha \wedge \neg\alpha$ assumption
2. α (1)
3. $\neg\alpha$ (1)
4. $\alpha \vee \beta$, (1), disjunction introduction
5. β, (3), disjunctive syllogism
6. $(\alpha \wedge \neg\alpha) \rightarrow \beta$ (5) by conditional proof, discharging (1).

It is clear that given $\mathcal{O}(\alpha \wedge \neg\alpha)$, $\mathcal{O}\beta$ is derivable only if $\alpha \wedge \neg\alpha$ and β are logically equivalent and logical equivalents can be substituted in deontic contexts via SLE or equivalent rules.

1.3.1.7 Normative Conflicts and Explosion

Contrary-to-duty cases introduced us to the well-known phenomenon of deontic explosion, namely the fact that in SDL (and some weaker logics), incompatible obligations generate arbitrary obligations.

The Multiple Operator Approach Normative conflicts are often caused by incompatible obligations issued by different sources: one natural idea to avoid conflicts is therefore to use multiple operators, one for each normative source. Each operator would fully take advantage of the core principles, without generating paradoxes.

Here is a very naive example: say the law imposes an obligation on you to do p, which is in turn forbidden by your religion. If we used only one operator, say, O, to express these requirements, we would formalize the situation in the following way: $Op \wedge O\neg p$. By agglomeration, we would derive $O(p \wedge \neg p)$, from which, by simple propositional logic, follows that everything is obligatory: Oq, with an arbitrary q.

If we used *two* different operators, say O_l and O_r, then we would have $O_l p \wedge O_r \neg p$. But at this point we could not derive, without further provisions, $O(p \wedge \neg p)$. No contradiction follows.

This was a rather crude example: the operators may be very finely indexed and ingeniously relativized: Castañeda made use of this approach to deal with deontic paradoxes (cf. Castañeda 1974, 1975, 1977). Moreover, there are very sophisticated contextualized deontic logics, indexed to different authorities, codes, and agents.[23]

The (naive) multiple operator approach, however, suffers from severe deficiencies.

Following Goble (2013c), I will just point out one of them, the most serious when we are concerned with logic: it is very hard to represent the conclusion of an argument with multiple operators.

[23] See for instance Goble (2005, 2009, 2013a, c).

Say the law requires you to do $p \vee q$, and your religion requires you to do $\neg p$, but says nothing on q.[24]

According to the naive multiple operator approach, we have $O_l(p \vee q)$ and $O_r(\neg p)$, but no way to adhere to our intuition that you ought to do q.

As a possible solution to this inconvenience, we might want to say that both the legal and the religious requirements are provisional, or prima facie, and the resultant, actual, or *all-things-considered* duty is to be established by some process, taking into account all prima facie duties, to see which one is stronger, or more important, etc.[25]

Prima facie versus All-Things-Considered Oughts This suggestion, rooted in the work of William D. Ross, takes us directly to the second strategy available to avoid normative conflicts and at the same time keeping core deontic logic principles.

The second strategy does not postulate different kinds of obligatoriness operator, standing for different oughts, but just one flavor declined in two degrees, so to speak: prima facie versus *all-things-considered* oughts.

A *first*, traditional account of prima facie versus *all-things-considered* oughts proposes to order those prima facie oughts according to some principle or some relative weight, and then consider as the resultant, *all-things-considered* ought that specific prima facie ought that outweighs all the others.

A *second* approach takes as the fundamental normative notion that of reason, in terms of which prima facie oughts are then analyzed: a fact is a prima facie ought if there is a normative reason for it. *All things considered* oughts are then undefeated prima facie oughts.

A *third* approach reverses the perspective: while in the first and second account *all-things-considered* oughts were derived, according to this third account they are fundamental, and prima facie oughts are defined in terms of *all-things-considered* oughts. Prima facie oughts are just *ceteribus paribus* principles about *all-things-considered* oughts, usually expressed through a defeasible, nonnormative conditional.

A *fourth* approach is akin to the first in considering *all-things-considered* oughts as derived from prima facie oughts, although over the latter it is defined a formal ordering.

There are also hybrid approaches.

Until now, we have only seen different approaches to *avoid* conflicts and keeping core principles of deontic logic.

But there are proposals to allow conflicts between oughts, and just to avoid disastrous consequences, such as universal obligatoriness or deontic explosion. How is that possible? Again there are two macro-approaches: one moderate, the other quite radical in changing completely the logic.

As for moderate approaches, these usually give up or modify some core principles of deontic logic. I follow Goble's order of exposition.

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[24] In this context it is not important whether this $\neg p$ is a forbiddance, or a negative obligation, or whatever; nor whether the fact q is ignored by your religion makes it permissible.

[25] The idea of prima facie and *all-things-considered* duties is traditionally traced back, in modern times, to Ross (1930).

First, there is a non-Kantian approach, i.e. an approach which does not accept the OIC (ought implies can) principle ($Op \rightarrow \Diamond p$). Normative conflicts become logically consistent, but deontic (or permissive) explosion is not avoided, because of principle K and the classical logic ECQ (*ex contradictione quodlibet*).

Second, there are non-aggregative systems, i.e. systems that do not accept agglomeration ($Op \wedge Oq \rightarrow O(p \wedge q)$). In so doing, normative conflicts are both consistent and do not generate deontic explosion. But the resulting logic, assuming all other standard principles of SDL, is non-normal. Goble calls this resulting logic **P** and goes on to argue that it is too weak to account for the plausibility of several arguments which are intuitively valid.

Third come non-distributive systems, which basically reject Necessitation (but often also OIC and the interdefinability of obligation and permission) and only keep agglomeration and substitution of logical equivalents. Again, while admitting normative conflict as logical and consistent and avoiding deontic explosion, these systems cannot fully explain the plausibility of certain plausible arguments.

Fourth, systems with limited replacement. These systems for the most part leave untouched the other core deontic principles, but they modify the substitution of logical equivalents. I will study one such system, which I call Hyperintensional Deontic Logic (HDL) at length in Chaps. 4 and 5.

1.3.2 Problems Highlighted in the Linguistic Literature

Kai von Fintel (in 2012) lists five main kinds of problems with classical semantics for deontic modals: (i) problems having to do with right upward monotonicity (Ross' Paradox, Professor Procrastinate), (ii) moral dilemmas, (iii) information sensitivity (Miners Paradox), (iv) the interpretation of certain deontic conditionals (such as if p, ought p), (v) issues surrounding the (non-)gradability of deontic modal expressions.[26] In what follows I will briefly consider (i), (ii) and (v) and add brief discussions of (vi) scope order, (vii) raising versus control verbs and (viii) weak necessity.

1.3.2.1 (Upward) Monotonicity

The first problem is whether deontic contexts are upward monotonic, that is, they license inferences of the form: if $\vdash \phi \rightarrow \psi$ then $O\phi \rightarrow O\psi$. Both Kripke's and Kratzer's approach license this pattern, which realizes a rule of SDL sometimes called *inheritance*.

Critics of monotonicity impute to it paradoxes like Ross's, Professor Procastinate, and others. An influential response on behalf of the classical approach is von Fintel's:

[26] Other accessible surveys are to be found in Cariani (2013) and Lassiter (2017), for instance.

deontic modals indeed are upward monotonic, as is suggested by negating modals and other phenomena in its vicinity. Cases like Ross' are best explained by free choice effects, and free choice inferences are best treated as implicatures, and not by breaking the monotonicity or coming up with a non-classical semantics.

Other cases where upward monotonicy seems jeopardized would have to be explained by context-shifts.

Standard counterarguments involve testing for implicatures (for instance via their cancellability, like those advanced by Cariani) or connecting the issue of monotonocity to other problematic features (Lassiter 2017). We shall see some of these replies in the next section.

1.3.2.2 Dilemmas

The resources of classical semantics are inadequate to express dilemmas, i.e. situations where one ought to ϕ and also ought $\neg\phi$. In fact, according to the classical semantics, ought ϕ is true iff all the best world in the modal base are ϕ-worlds. But for ought $\neg\phi$ to be true, all the best world in the modal base are $\neg\phi$-worlds. This is impossible, and ought to ϕ and ought $\neg\phi$ cannot be true given the same conditions. A standard solution is to require a small modification in the semantics in order to adopt the so called *disjunctive approach*, where neither ought to ϕ and ought $\neg\phi$, but ought to $(\phi \vee \neg\phi)$ is licensed.

We have seen that, otherwise, there would be a deontic explosion.

Although there are no worked-out solutions—as far as I am aware—within the classical semantics literature, von Fintel (Von Fintel 2012, Sect. 3) points out that moral dilemmas could be accommodated once one lets the classical semantics have different (sub)ordering sources. This seems a way to reconcile this problem to von Fintel's own proposal to deal with weak necessity (i.e. 'ought' and 'should', rather than 'must' and 'have to') which would then be able to model prima facie obligations, which admit of conflicts, rather than full-fledged, all-things-considered ones.

1.3.2.3 Gradability

Are modal notions all-or-nothing or rather scalar notions? Standard approaches in logic and semantics have dealt with modality as a non-scalar notion, notably by interpreting modalities as quantifiers, which are clearly non-scalar concepts. If deontic modals were in fact gradable, their semantics would need a serious adjustment, given that quantifiers are not gradable.[27]

[27] For an inquiry on the scale of practical reasons, see Chap. 6. Faroldi and Ruiz (2017) tackle the problem of inconsistent linguistic data to build a coherent scale of moral adjectives by arguing that they are multidimensional.

The hypothesis that deontic modality is somehow scalar is supported by the interaction and acceptability of (some) deontic modal terms with degree modifiers (completely, fully, totally), comparatives, and superlatives.

This insight is not, however, new. We have seen that a certain version of ordering semantics is equivalent to the standard Kripke-Kratzer one. But to a certain extent, orderings can be associated to a measure we could use to give a semantics to gradable terms. In the deontic case we have already seen a certain connection between the ordering on worlds and goodness. But as Lassiter (forthcoming) appropriately notes, $\mathcal{O}$ and $\mathcal{P}$ are not given a scalar semantics, and the theory cannot explain their gradability, although their semantics is based, at least partially, on a scalar structure.

For instance, Lewis (1973), Chap. 5 suggests a strategy for 'good' more or less like the following (paraphrased): let W be the set of worlds, D a set of degrees, $\leq$ a total preorder (i.e., connected, reflexive, transitive) on D, a measuring function $f : W \to D$.

But this lets us compare the goodness of worlds, not of propositions.

Modulo the limit assumption, one version of Kratzer' system is provably equivalent except for totality: there can be incomparable worlds.

A "lifting" trick which considers the rank of the best worlds in each proposition (assuming of course that propositions are sets of worlds) is needed:
a is at least as good as b iff, for some w in a, $f(w') \leq f(w)$ for all w' in b.

A slightly more elegant lifting is proposed and defended by Goble (1996, 2003) and Lassiter (2011), where each proposition is assigned a degree of goodness resulting from the weighted average of the worlds where the proposition is true, and each world has a weight equal to the probability it is actual, conditional on the probability the proposition is true.

The resulting scale is intermediate, rather than additive. To make a rough comparison, probability is associated with additive scales, whereas expected utility value is associated with intermediate scales. Height is additive, temperature is intermediate. Epistemic concepts should be modeled additively, deontic ones intermediately.

This predicts, in turn, rather interesting patterns of inference over disjunction.

As Lassiter notes, in fact, additive scales are closed under a join-like operation (such as disjunction), whereas intermediate scales are not. This solves some issues attributed to monotonicity like Ross's paradox.

But this is just the tip of the iceberg.

There has been much work recently on (deontic) modals and probability. Some theorists tried to give a semantics for 'ought' and 'should' relativized to probabilistic states.

Lassiter (in Lassiter (2011, 2017), among other works) mounts a forceful and resourceful defense of the scalar nature of modality, and also puts forward a proposal.

Lassiter develops, in analogy to that given to gradable adjectives, a scalar semantics using a fully closed additive scale, which is provably equivalent to a finitely additive probability measure. The basic idea is that the semantics of modal expressions is best cashed out in terms of functions mapping propositions to a scale and a threshold value.

As for deontic modals, Kratzer's theory is confronted with five sets of problems. Lassiter thinks his approach can either solve or deal with them markedly better.

First, according to Lassiter, deontic (and bouletic, although my discussion here is limited only to deontic) modals are not upward monotonic, which Kratzer's quantificational semantics predicts them to be. Upward monotonicity licenses the following inferences, which are intuitively invalid:

1. $\mathcal{O}\alpha \rightarrow \mathcal{O}(\alpha \vee \beta)$
2. $\mathcal{O}(\alpha \wedge \beta) \rightarrow \mathcal{O}\alpha$
3. $\mathcal{O}(\alpha \wedge \beta) \rightarrow \mathcal{O}\alpha \wedge \mathcal{O}\beta$

(1) is the cause of Ross' paradox, (2) of professor Procastinate cases, (3) of various bad instances of agglomeration, such as Jackson's chicken.

Second, Kratzer's semantics fares badly in situations where information sensitivity is crucial, such as the Miners' case, where in general the intuitive obligatory outcome is not the optimal one, but is suboptimal, although it is optimal given the relevant circumstances.

Third, the interaction of deontic modality and comparatives (and equatives) is poorly handled by Kratzer's semantics: first, because many instances of deontic modality can form comparatives and equatives; second, because it seems that deontic modals come in more degrees of strength than standard quantifiers (basically, ought is a weak necessity modal hardly captured by either the universal or particular quantifier analysis).

Fourth, the standard complaint against Kratzer's ordering of worlds, which, being a partial (or pre) order, is not total, and therefore leaves a lot of worlds incomparable, even when they are intuitively comparable.

Fifth, Krazter's theory rules out deontic conflicts. As a matter of fact there seem to be many situations where conflicts of oughts are in force; however, Kratzer's semantics makes conflicts false. There may or may not be real deontic conflicts, but it is not a semantics task either to limit their expressibility or rule out their (possible) existence.

Lassiter's solution is fairly simple, from a technical point of view, as he proposes to move away from ordinal scales (used in ordering of worlds by standard semantics) towards interval scales, given that the latter lets us measure quantitively. Therefore, a deontic modal is true if it is mapped to a point of the scale exceeding a threshold value determined by the lexical semantics of that modal and the context ('require', 'have to', 'must', 'need' will have a higher threshold value than 'want', 'ought', 'should', 'supposed to'; whereas 'may', 'permitted', 'allowed' will have a very low threshold (p. 165)).

Lassiter (2011) semantic proposal considers "ought ϕ" true iff the expected value of ϕ is *significantly* higher than the average expected values of the relevant alternatives. He claims he can accomodate ought conflicts, since it may well be the case that two (incompatible) prejacents have a higher value than the lowly valued relevant alternatives.[28]

[28] On expected value analysis of deontic modals, see Cariani (2014), among others.

How does such a theory impact on the aforementioned problems? *First*, such a semantics is not upward monotonic (because of the nature of the scale Lassiter uses, intermediate rather than additive). *Second*, probability measures obviously (at least partially) contribute in solving some information sensitivity issues. *Third*, grammatical gradability is much more adequately accounted for with this sort of scalar semantics. The *fourth* point is bypassed structurally employing a probability space. *Fifth*, in the worlds of von Fintel: "Lassiter (2011) offers his account as one that makes space for conflicting oughts. Ought ϕ is true iff the expected value of ϕ is significantly higher than the average expected value of all the relevant alternatives. This allows two incompatible prejacents that both ought to be true: as long as there are plenty of alternatives with very low expected value, two alternatives can be significantly higher than the average of all of them (Von Fintel 2012)" who is, however, quite critical of Lassiter's proposal, on the grounds that what he treats as dilemmatic cases are not dilemmas at all.

Since Lassiter's proposal has changed quite a bit from Lassiter (2011) to Lassiter (2017), and we cannot go through all the details, I will just offer a couple of very general remarks more philosophically than technically motivated.

Assuming Lassiter's proposal technically works, i.e. the problems he identifies for the standard paradigm are indeed problems, and his solutions are sound, there remains a very general philosophical objection to his probabilistic account, an objection we could, roughly, assimilate to a very subtle form of "factuality" or the $p \rightarrow \mathcal{O}p$ problem (*ab esse ad debere (*non*) valet consequentia*).

Roughly, "$\mathcal{O}p$ is true" iff its expected value is significantly higher than the expected values of the alternatives. The expected values are, roughly, an expression of the (quantitative) degree of desirability, goodness, etc. of the alternatives measured as their probability to be actual. But this is a factual (or epistemic) consideration, and has nothing to do with normativity. The fact that the chance of universal happiness is 0 has prima facie nothing to do with the fact that we ought (not) to promote universal happiness unless we build explicitly in our normative theory the idea that obligations should be effective, and there is no point in having a useless, 0-chance, obligation. But this is not a trait we want to build in our semantics.

Another objection to Lassiter's account of deontic modals is the following: the fact that the threshold value is fixed by context, albeit in a certain given range (almost maximal for 'must', 'have to', almost minimal for 'may'), makes the semantics of (deontic) modal expressions structurally dependent on pragmatics, in a way Krazter-style approaches do not. The contextual parameters in Kratzer-style approaches provides background information, i.e. basically which worlds are accessible, in the domain of quantification, and their relative ranking, but their structural meaning, so to speak, remained independent from this context-supplied information: necessity modals are universal quantifiers, possibility modals are particular quantifiers. Lassiter proposal, instead, makes them structurally dependent on context, because the threshold value is as a matter of fact provided by the context. I do not know if it is possible that in two different contexts mid-range modals are assigned threshold values so low, or so high, to be able to mean respectively possibility or full-scale necessity modals.

Note that Lassiter's is, in some ways, a situationist semantics, because the order is imposed on propositions, rather than worlds. Is it still in a strong sense a possible world semantics? The downlifting of the ordering of worlds to an ordering on propositions is obtained by "assigning to each proposition a degree of goodness which is a weighted average of the goodness of the worlds in it, where each world receives a weight equal to the probability that it will be true if the proposition occurs" (cf. Lassiter forthcoming).

Lastly, a problem regarding logical equivalents.

The standard Kolmogorov axioms for probability imply that logically equivalent propositions are equiprobable. Given how Lassiter employs probability measures in his semantics, it is not clear that he can distinguish propositions which are logically equivalent but normatively "different".

In Sect. 4.5.2 I will try solve this problem within a scalar semantics.

1.3.2.4 Scope Order

From a formal point of view, the classical set-up allows for the iteration of (differently flavored) modals with no particular restrictions.

But some typological crosslinguistical data show (cf. Nauze 2008) that the following general scope order may recur: epistemic modality > participant-external modality > participant-internal modality. In other words, epistemic modals outscope deontic and goal-oriented modals, which outscope ability and need-related modality. A straightforward consequence, if this is true, is that epistemic modals cannot be embedded in deontic modals.[29]

If this general scope order is true, Nauze says, then the standard Kratzerian approach is in trouble, for it cannot explain this restriction, occurring in all (most?) natural languages, with its semantic resources.

The Kratzerian might say the reasons for this restriction are to be found at the syntactic level. This reply is unconvincing for two reasons: *first*, as Nauze (2008), p. 178 points out, there is no unifying syntactic analysis able to explain all the data. Even the most powerful and obvious one, in terms of raising versus control verbs (see next paragraph), is at best simplistic, if not wrong: first it should predict that, say, deontic modals cannot embed under epistemic ones because the former are control, the latter raising, and control verbs cannot embed under raising ones. But one premise is false (since some deontic modals are raising), and the thesis does not follow (therefore epistemic modals should embed under some deontic modals); second, the whole raising versus control analysis may be false, given a strong argument for the thesis that all English modals are somewhat raising (Wurmbrand 1999). Third, this analysis, even if true, would be able to account only for those languages with a raising/control syntactic distinction, a distinction some languages (languages preserving the scope order) lack.

[29]Naturally one finds several studies of scope order in the linguistic literature. Here I focus only on the issue of epistemic > deontic but not viceversa, which seems to be shared by most theorists.

Second, assuming the scope order isolated by Nauze is the case, it cannot be explained (away?) syntactically, because it seems too consistently crosslinguistic to be a feature of the meaning of the respective modal categories, and therefore to be explained by the semantics.

Why can't the standard framework explain scope order semantically? Assume there is a sentence with two modals, m_1 and m_2, such that one is in the scope of the other, say $m_1 > m_2$. Under the usual compositional machinery, the Kratzerian standard approach would predict that there are two pairs of ordering sources and modal bases, each relative to m_1 or m_2. But there is no constraint whatsoever in this kind of modal base to influence the scope order, since modal bases deal with different kinds of modality in a uniform way.

But von Fintel apparently came up with some possible counterexamples (Morphology, Syntax, and Semantics of Modals, pp. 15–16): "An insurance company will only pay for an expensive test if there is a possibility that the patient may have Alzheimer's. (43) For the test costs to be reimbursed, it has to be (deontic) possible (epistemic) that the patient has Alzheimer's. Or: We are visiting an English mansion and it so happens that a murder takes place while we are there. The police determine that the culprit is either a certain crazy tourist that was there that day or the victim's ex-lover. You are disappointed that while you find yourself in a real-life English mansion with a real-life murder, the usual detective-story suspect, the butler, is already exonerated and you say: (44) It ought to (deontic) be possible (epistemic) that the butler did it."

1.3.2.5 Raising versus Control Verbs

The question of raising versus control verbs and modality is whether (and if so, why) there is a certain distribution between flavors of modality and kinds of modal verbs either raising or control.

For the purposes of this chapter, even simplifying a bit, in raising verbs the thematic subject is different from the grammatical subject, which is raised at the beginning of the sentence. Consider the following:

(48) It seems (that) John is tall

The grammatical subject of 'seems' is 'it', but the thematic subject is in fact John.

In control verbs, at least to a first approximation, the thematic subject coincides with the grammatical subject:

(49) John ought to block neither shaft

(49) ascribes a certain (deontic) property to the subject, John. Raising versus control phenomena are quite common in English and other languages, but lacking in others.

Shared consensus was that English epistemic modal verbs are all raising, and English deontic modal verbs are all control. If this thesis were true, there would be a principled (albeit syntactic) reason to explain scope order, i.e. to explain why, for instance, epistemic modals seem to outscope deontic modals, but not viceversa.

Unfortunately this simple picture is almost certainly wrong. First of all, many languages lack a raising versus control distinction, so that the appeal to a syntactic feature to explain a crosslinguistic, semantic phenomenon is dubious. Second, there are counterexamples to the hypothesis that deontic modals are just control verbs.

But since the scope order hypothesized by Nauze might be wrong, given at least the plausible counterexamples of v. Fintel and others, the issue of raising versus control modal verbs may be periferal with respect to the semantics of modality.

1.3.2.6 Weak Necessity

Some believe there is something like a modality, intermediate in strength between (strong) necessity, expressed in English by modals like 'must' and semi-modals like 'have to' and possibility, expressed in English by modals like 'may' and 'can'. Such "weak" necessity would be expressed, in English, by 'ought to' and 'should'.

I will skip, for the time being, the question of whether this is a real modality or is just a quirky trait of the English language in order to express what other languages (such as Romance languages) express using a different mood, namely, the conditional mood in main clauses.

So, let us assume that weak necessity does in fact exist.

How to model the concept of weak necessity using classical possible world semantics? Traditionally, given the correspondence of "for all" to necessity and of "some" to possibility (reduced to quantification over accessible worlds), it is quite natural to think of modelling weak necessity using another natural language quantifier: "most". In this way, "you should not steal" is true at w iff for *most* w', wRw' and "you don't steal" is true at w'. But given that some (most?) models have non-finite cardinality, it is impossible to know when "most" accessible worlds satisfy the requisite, and so when, if ever, weak necessity modals are true.

Another intuitive proposal to account for natural language (and conceptual?) differences between strong and weak necessity modals would be to multiply the accessibility relations, for which the very influential Von Fintel (2008) is the *locus classicus*. But this is potentially uninformative, because just one modal can be compatible with many different accessibility relations.

Narrog (2012), p. 251 reports that of 200 genealogically distinct languages, only approximately 60% have weak (deontic) necessity, and only approx. 30% have grammaticalized means to express it. Rubinstein (2014) also discusses some data from modern Hebrew, which expresses weak necessity neither lexically (cf. English 'ought') nor compositionally (cf. Romance languages conditional mood), but rather employing the language of (evaluative) comparison (preferable, better/best, fitting, worthwhile).

This is taken as evidence that weak necessity, rather than being expressible in the usual quantificational semantics just as strong necessity and its dual, possibility (permissibility), genuinely needs a comparative semantics. Rubinstein further argues that a comparative semantics for weak necessity does not clearly warrant a switch to a probabilistic framework, as several theorists would have it. In fact, Rubinstein tries

to show that a traditional quantificational semantics can serve both comparative (for weak necessity) and non-comparative (for strong necessity) purposes, depending on the treatment of the ordering source. Rubinstein (2012) puts forward the hypothesis that whereas strong necessity modals reference priorities which are presupposedly shared, weak necessity modals make reference to a different, more restricted, domain of those commitments that are not shared by everyone.

Another reason to sharply distinguish strong and weak necessity is, at it is well known from Horn's writings (cf. for one Horn 1989), their different behavior with regard to neg-raising. Neg-raising verbs allow the interpretation $\mathcal{V}\neg\phi$ from the original utterance $\neg\mathcal{V}\phi$ (an example: "I do *not* think he is lying" is usually interpreted "I think he is *not* lying").[30]

There are several proposed explanations: from scalar implicatures (Romoli 2013) to presuppositions (see Gajewski 2007 and Heim and Kratzer 1998 and see also Horn 1989). Bartsch (1973) proposed instead to explain away neg-raising with a simple and elegant hypothesis, namely, of an excluded middle inference of the form:

1. $\neg\mathcal{V}\phi$ ass.
2. $\mathcal{V}\neg\phi \vee \mathcal{V}\phi$ ass.
3. $\mathcal{V}\neg\phi$ (1), (2), PC

It would be interesting to apply these tests also to possibility and weak possibility, to investigate whether there is such a thing. From an historical linguistic perspective, in fact, 'might' is the past/subjunctive form of 'may', precisely as 'ought' of 'owe', or 'should' of 'shall'. Now, given that both 'ought' and 'should' express, in contemporary English, weak necessity, one could ask, at least on theoretical grounds, whether 'might' could express weak possibility. Additional evidence comes from the compositional way of expressing weak necessity in Romance (and other) languages by using the conditional mood: as there is a self-standing form of necessity modals in the conditional mood used to express weak necessity, so there might be a self-standing form of possibility modals that could be used to express weak possibility, if there is such a thing.

And indeed there is a self-standing form of alleged weak possibility modals in the conditional mood in Romance languages, for instance the Italian conditional 'potrei' (I may-COND 1SG).

Three observations: First, I am not sure, without further investigation, that 'potere' in the conditional mood has deontic flavor at all, in positive form (i.e. not in asking permission, as in "Potrei aprire la finestra?" "May-COND 1SG (I) open-INF-PR the-F window?"). This feature might be present also in contemporary English, where a

[30] As an aside, I note here the semantic classes of neg-raising predicates individuated by Horn (1989), p. 323 (and see also Rubinstein 2014): opinion ('think', 'believe', 'suppose', 'imagine', 'expect', 'reckon', 'feel'), perception ('seem', 'appear', 'look like', 'sound like', 'feel like'), probability ('be probable', 'be likely', 'figure to'), intention/volition ('want', 'intend', 'choose', 'plan'), Judgment/(Weak) obligation ('be supposed to', 'ought', 'should', 'be desirable', 'advise', 'suggest'). Note that, in English, verbs like 'desire' and 'hope', almost synonymous to 'want', are not neg-raising. Many of these predicates are plausibly hyperintensional. As far as I know, any direct link has yet to be investigated.

rare deontic use of 'might' is in reporting a past permission. But in Italian, especially in the second or third person, there is plausible ground to maintain that there is a deontic reading also in a positive form, as in "Potresti fare querela" "(You) might-COND 2SG file-INF PR a formal complaint". Second, we should run the usual test of "weakness". The most common one is the implication pattern: as 'must' implies 'ought', or 'ought' implies 'not have to', so 'may' should imply 'might', or 'might' should imply some sort of external negation of 'may'. Third, if it is true that weak modals are neg-raising, whereas strong modals are not, then we should also run the neg-raising test.

If there were in fact such a thing as weak possibility, and further assuming that (weak) modal concepts are best understood in a scalar semantics context, there would be an interesting question about what kind of scale 'might' might have, i.e. lower-bounded, open, intermediate etc., and the question would be enlightened by the hypothesis of a symmetrical scale with regard to the one ought has, i.e. if 'ought' were upper-bounded, 'might' should be lower bounded, etc. This opens the following scenarios: if there is such a thing as weak possibility, which would be expressed by 'might', then the first hypothesis is that 'might' and 'ought' could be dual, exactly as strong necessity and possibility are; whereas the second hypothesis is that 'must' (strong necessity) is the dual of weak possibility ('might', 'could'); 'ought' (weak necessity) is the dual of strong possibility ('may', 'can').

As far as I know, no research has been done on this topic. The logic of explicit permission and obligation (where two implicit, "weak" duals can be defined) developed in Anglberger et al. (2016) and reported in Chap. 4 could be interpreted in this sense.

Syntactic Ways To Model Different Obligatoriness Strength Leaving semantics aside, can we find the resources we need to express the difference in strength between 'must' and 'ought' in the syntactic part of our (logical) language?

There are at least two ways of doing so: (i) with a unique operator for obligatoriness; (ii) with two different operators for obligatoriness.

On the single operator approach, one would just iterate it to express something stronger: Oa would be weaker than OOa, whereas $OOp \wedge Op$ would be stronger than $Op \wedge \neg OOp$. (For one, Hansson 2013, p. 451n considers the single operator approach "crude" and a "dubious way to express strength.")

On the multiple operators approach, one would just have an operator for each "degree" of obligatoriness. Let us suppose we have just two degree of obligatoriness, one expressed by 'must' in English, the other expressed by 'ought'. We would have two operators O_1 and O_2, such that if for any sentence p, $O_1 p$ entails $O_2 p$, but it is not the case that $O_2 p$ entails $O_1 p$, then O_1 is stronger than $O_2 p$.[31] In an informal way, every 'must' entails an 'ought', but not every 'ought' entails a 'must': 'You

[31] More formally and precisely, one should define that O_1 includes O_2 iff for every sentence q $O_2 q$ entails $O_1 q$, and O_1 properly includes O_2 iff O_1 includes O_2, but O_2 does not include O_1. Proper inclusion would then be a guide for difference in strength. Hansson (2013) follows this strategy, and furthermore defines a set $\mathbb{O}$ of obligatoriness operators inclusion-complete when for every $O_1, O_2 \in \mathbb{O}$ either O_1 includes O_2, or O_2 includes O_1.

must listen to *Simon Boccanegra*' entails 'You should listen to *Simon Boccanegra*'; but it is not the case that 'You should see the last production of *Fidelio*' entails 'You must see the last production of *Fidelio*', under the assumption that 'must' is in fact stronger than 'ought'.

Must/Ought as Extreme/Non-extreme Words Portner and Rubinstein (forthcoming) maintain that weak modals like 'should' and importantly differ from strong modals like 'must' and adjectives like 'crucial' in the same way that non-extreme expressions ('pretty') differ from extreme ones ('gorgeous'). What are extreme expressions? In the case of adjectives, "Extreme adjectives have a number of properties which show that they have scale-based meanings which are intimately related to the semantics of non-extreme expressions, but they are also distinct both semantically and pragmatically from ordinary gradable adjectives (p. 11)."

The phenomenon of extreme expressions, if true, seems to run parallel to the positive/comparative/superlative degree of adjectives, for instance in languages, like Latin, an adjective like 'certus' at the positive degree has a comparative form, 'certior', and a superlative form, 'certissimus'.

But certain adjectives, such as 'bonus' ("good"), 'malus', 'magnus', 'parvus', instead of inflecting with the comparative (superlative) suffix the same root, just change the word employed for the comparative and the superlative. For example, the comparative of 'bonus' ("good") is 'melior', of 'malus', 'peior', of 'magnus', 'maior', of 'parvus', 'minor'; and their superlatives are respectively 'optimus', 'pessimus', 'maximus', 'minimus'. (Cf. English good, better, best, etc.)

The idea is certainly intriguing, but the historical-linguistic evidence is not clear, yet.

Where we are so far So far we have seen a logic, with a syntax and semantics, and several problems this package entails—especially when a comparison with natural language data is made. But what is our purpose? Should we merely describe natural language? Should we just aim at formal systems, which at best approximate certain aspects of practical reasoning sometimes expressed by explicit deontic language? One might then have the impression that different theories, rather than having substantial disagreements, are just talking past each other. It is also for this reason that in the following I assume the vantage point of the philosophical logician, rather than that of a linguist or semanticist; my aim will be to work with a suitable regimentation of natural language, rather than its faithful description. Let us hope that if there is a disagreement, it is a productive disagreement.

The problems briefly highlighted in Sect. 1.3 have elicited reactions within Kratzerian orthodoxy but have also caused a number of nonstandard proposals both in logic (where new research is quite fast-paced and much more fractured among different sub-communities) and in (formal) semantics, where in the past few years a smaller number of (somewhat new) frameworks emerged. As a brief conceptual map, we could enumerate existing proposals as follows:

- those involving a gradable or scalar dimension, with or without probability measures of some sort[32];
- contrastivist approaches[33];
- other proposals,[34] among which inquisitive[35] and dynamic semantics.[36]

In the past few sections, we have seen, however informally and cursorily, both an introduction to the classical account to the logic and semantics of deontic modals (Sect. 1.2), some well-known problems highlighted in the logic or the linguistic literature (Sect. 1.3), and very briefly, some new proposals and theories that have been put forward in the past few years.

To conclude this section in the same informal tone, let me note a trait (problem?) most of these theories have in common—this will also give us a glimpse of what is to come in the rest of this work.

Most of these theories suffer from what we may call the irrelevance problem.

Consider the following example:

(50) You ought to drive.

(51) You ought to drive or to drive and drink.

The prejacents of (50) and (51) are logically equivalent, because A is logically equivalent to $A \vee (A \wedge B)$.

But (50) and (51) cannot be considered equivalent obligations, as many legal systems take care to remind us. The fact that driving and (driving or (driving and drinking)) have the same truth-value in all situations is not enough to make them interchangeable in ought-contexts, preserving all normatively relevant features.

Consider symmetric predicates, like 'shake hands', 'get engaged'. Logically equivalent propositions employing them would stop being equivalent when put in the scope of a deontic modal.

No doubt some of theories considered in this chapter have the resources to rule out one or the other with some ad hoc moves. But what is going on, I surmise, is both deeper and more interesting than a few adjustments here and there. What is going on, I will argue in the rest of this work, is that deontic contexts are more discriminating than mere truth, or even necessary or logical truth. Deontic contexts are not extensional, and not even intensional. Prima facie, it seems that deontic contexts should not be closed under the substitution of logically equivalent contexts. Semantically, logical

[32] Among which, notably, Cariani (2014), Finlay (forthcoming, 2014), and Lassiter (forthcoming, 2011, 2017).

[33] Among which, notably, Cariani (2013) and Snedegar (2013, 2014).

[34] Such as the two-dimensional semantics of Chrisman (2015), on which see also Woods (2016) for an interesting review.

[35] For an introduction to inquisitive semantics, see Ciardelli et al. (2013); Ciardelli (2015); for an application to deontic modals Aher (2013); Aher and Groenendijk (2015), and to imperatives Aloni and Ciardelli (2016).

[36] See Mastop (2005, 2011, 2012) on dynamic interpretation of imperatives (or commands as imperative speech-acts) and Starr, forthcoming for deontic modals properly (Nauze 2008, Chap. 5), Silk (2013, 2015, forthcoming), and Willer (2014).

equivalence is not a good criterion for synonymity as far as deontic contexts are concerned. Does this mean that all hope is lost, and there is no possibility of a deontic logic, since, in a sense "truth is not enough"? The next chapter discusses the extensionality-intensionality-non-intensionality business in more precise terms, in order to gain some insights and apply them back to the deontic case, which will be done specifically in Chaps. 4 and 5.

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